Ch 4 Constant Mean Curvature Surfaces

(4.1 Omitted)

4.2 First notions in minimal surface

Def • Let $X : D \subset \mathbb{R}^2 \longrightarrow M \subset \mathbb{R}^3$ be a coordinate patch.

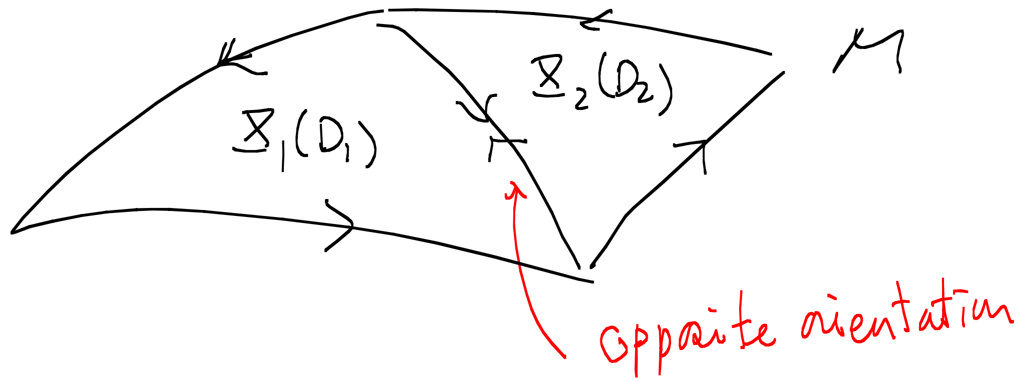
Then the area of the patch is

$$A_X = \iint_D |X_u \times X_v| \, du \, dv$$

• If $M = \bigcup_\alpha \overline{X_\alpha(D_\alpha)}$ such that if $\overline{X_\alpha(D_\alpha)} \cap \overline{X_\beta(D_\beta)} \neq \phi$, then they only intersect along boundary curves with opposite orientation, then area of M is defined by

$$A(M) = \sum_\alpha \iint_{D_\alpha} \left| \frac{\partial X_\alpha}{\partial u} \times \frac{\partial X_\alpha}{\partial v} \right| du\, dv$$

where $X_\alpha : D_\alpha \to M$ are coordinate patch



$H = 0$

<u>Fact</u> : Area minimizing $\Rightarrow$ minimal (see next section)
(wrt a fixed boundary)

<u>Recall</u> : For a Monge patch $X(u,v) = (u, v, f(u,v))$

$$\begin{cases} X_u = (1, 0, f_u) \\ X_v = (0, 1, f_v) \end{cases}$$

$$\begin{cases} \mathbf{X}_{uu} = (0, 0, f_{uu}) \\ \mathbf{X}_{uv} = (0, 0, f_{uv}) \\ \mathbf{X}_{vv} = (0, 0, f_{vv}) \end{cases}$$

$$(g_{ij}) = \begin{pmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{pmatrix} \quad \left(1^{st} \overset{(\text{metric})}{\text{fundamental form}}\right)$$

$$(h_{ij}) = \frac{1}{\sqrt{1+f_u^2+f_v^2}} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix} \quad \left(2^{nd} \text{ fundamental form}\right)$$

$\therefore$ Mean Curvature $H = \dfrac{(1+f_v^2) f_{uu} - 2 f_u f_v f_{uv} + (1+f_u^2) f_{vv}}{2 \, (1+f_u^2+f_v^2)^{3/2}}$

Prop 4.2.1 $M = \{(u, v, f(u,v))\}$ is minimal

$$\Longleftrightarrow \boxed{(1+f_v^2)f_{uu} - 2f_u f_v f_{uv} + (1+f_u^2)f_{vv} = 0}$$

Minimal Surface Equation

Recall that a non-planar minimal surface of revolution must be a catenoid. Correspondingly, we have

Theorem 4.2.2 (Catalan's Theorem) Any ruled minimal surface in $\mathbb{R}^3$ is part of a plane or a helicoid.

(Pf: Omitted.)

4.3 Area Minimization

Suppose $M: z = f(x,y)$ = surface of least area with boundary C.

Let D = domain of f with boundary curve ∂D such that "$f(\partial D) = C$".

Consider the family of nearby surfaces of the form

$$M^t: \quad z^t(x,y) = f(x,y) + t\,g(x,y) \quad , \ |t| \text{ small},$$

where g is a function defined on D such that

$$g\big|_{\partial D} = 0 .$$

A Monge patch for M^t is given by

$$\mathbf{X}^t(u,v) = (u, v, f(u,v) + t\,g(u,v)).$$

Then $\begin{cases} X_u^t = (1, 0, f_u + t g_u) \\ X_v^t = (0, 1, f_v + t g_v) \end{cases}$

$$\Rightarrow \quad |X_u^t \times X_v^t| = |(-(f_u + t g_u), -(f_v + t g_v), 1)|$$

$$= \sqrt{1 + (f_u + t g_u)^2 + (f_v + t g_v)^2}$$

$$= \sqrt{1 + f_u^2 + f_v^2 + 2t(f_u g_u + f_v g_v) + t^2 (g_u^2 + g_v^2)}$$

$$= \sqrt{1 + |\nabla f|^2 + 2t \langle \nabla f, \nabla g \rangle + t^2 |\nabla g|^2}$$

$$\therefore \quad \underset{\displaystyle A(t)}{\overset{\displaystyle A(M^t)}{\|}} = \iint_D \sqrt{1 + |\nabla f|^2 + 2t \langle \nabla f, \nabla g \rangle + t^2 |\nabla g|^2} \, du \, dv$$

$$\Rightarrow \quad \frac{dA}{dt} = \iint_D \frac{\langle \nabla f, \nabla g \rangle + t |\nabla g|^2}{\sqrt{1 + |\nabla f|^2 + 2t \langle \nabla f, \nabla g \rangle + t^2 |\nabla g|^2}} \, du \, dv$$

Since M is area minimizing (wrt the fixed boundary C),

$$\frac{dA}{dt}(0) = 0$$

i.e.

$$\boxed{\iint_D \frac{\langle \nabla f, \nabla g \rangle}{\sqrt{1+|\nabla f|^2}}\, du\, dv = 0}$$

or

$$\iint_D \left[\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}} \right) g_u + \left(\frac{f_v}{\sqrt{1+|\nabla f|^2}} \right) g_v \right] du\, dv = 0$$

Green's thm $\Rightarrow$ (since $g|_{\partial D} = 0$)

$$\iint_D \left[\frac{\partial}{\partial u} \left(\frac{f_u}{\sqrt{1+|\nabla f|^2}} \right) + \frac{\partial}{\partial v} \left(\frac{f_v}{\sqrt{1+|\nabla f|^2}} \right) \right] g\, du\, dv = 0$$

Since g is $\underbrace{\text{arbitrary}}_{\text{function with } g|_{\partial D} = 0}$, we must have (see ex 7.1.5 of Oprea)

$$\boxed{\frac{\partial}{\partial u}\left(\frac{f_u}{\sqrt{1+|\nabla f|^2}}\right)+\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{1+|\nabla f|^2}}\right)=0}$$

Expanding the equation :

$$\frac{f_{uu}}{\sqrt{1+|\nabla f|^2}}-\frac{f_u(f_u f_{uu}+f_v f_{uv})}{(1+|\nabla f|^2)^{3/2}}+\frac{f_{vv}}{\sqrt{1+|\nabla f|^2}}-\frac{f_v(f_u f_{uv}+f_v f_{vv})}{(1+|\nabla f|^2)^{3/2}}=0$$

$$\Longrightarrow \quad \boxed{(1+f_v^2)f_{uu}-2f_uf_vf_{uv}+(1+f_u^2)f_{vv}=0}$$

the minimal surface equation.

Hence we have

Thm (Thm 4.3.4 of Oprea)

If M is area minimizing (wrt a fixed boundary), then M is minimal.

4.4 Constant Mean Curvature

Let M = compact oriented immersed surface in $\mathbb{R}^3$

(immersed : may have self intersection,
but $X_u \times X_v \neq 0 \ \forall$ points on M)

$$\nu = \frac{X_u \times X_v}{|X_u \times X_v|} \quad \text{unit normal}$$

Note that M may not be a graph of a function, therefore, perturbation of M need to take the following more general form (on each patch):

$$M^t : \ Y^t(u,v) = X(u,v) + t V(u,v)$$

where V is a vector field on M $\left(V(u,v) \in \mathbb{R}^3 \right)$

Then the area $A(t)$ of M^t is

$$A(t) = \iint |Y^t_u \times Y^t_v|\, du\, dv$$

(more precisely
$$A(t) = \sum_D \iint_D |Y^t_u \times Y^t_v|\, du\, dv$$
sum over domains of the patches X.)

$$= \iint |(X + tV)_u \times (X + tV)_v|\, du\, dv$$

$$= \iint |X_u \times X_v + t[(X_u \times V_v) + (V_u \times X_v)] + t^2 V_u \times V_v|\, du\, dv$$

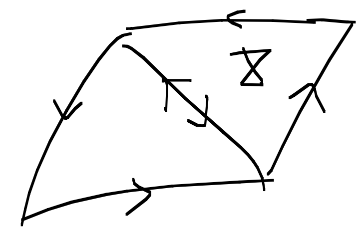
$$= \iint \left[|X_u \times X_v|^2 + 2t\langle X_u \times X_v,\ X_u \times V_v + V_u \times X_v\rangle + O(t^2)\right]^{1/2} du\, dv$$

$$\therefore\ A'(0) = \iint \frac{2\langle X_u \times X_v,\ X_u \times V_v + V_u \times X_v\rangle}{2\left(|X_u \times X_v|^2\right)^{1/2}}\, du\, dv$$

$$= \iint \langle U,\ X_u \times V_v - X_v \times V_u\rangle\, du\, dv$$

$$= \iint \left[\langle V_v, U \times \mathbf{x}_u \rangle - \langle V_u, U \times \mathbf{x}_v \rangle\right] du\, dv$$

Ex. 4.4.1 of Oprea $\Rightarrow$ On each patch $\mathbf{x}$



$$\iint \left[\langle V_v, U \times \mathbf{x}_u \rangle - \langle V_u, U \times \mathbf{x}_v \rangle\right] du\, dv$$

$$= -\iint \langle V, 2H\, \mathbf{x}_u \times \mathbf{x}_v \rangle\, du\, dv$$

$$- \int_{\partial} \left[\langle V, U \times \mathbf{x}_u \rangle\, du + \langle V, U \times \mathbf{x}_v \rangle\, dv\right]$$

$\uparrow$ boundary, where H = mean curvature of M.

Since M is cpt, oriented, line integrals along the boundaries cancel each other. Therefore,

$$A'(0) = -\iint \langle V, 2H\, \mathbf{x}_u \times \mathbf{x}_v \rangle\, du\, dv$$

$$= -\iint 2H \langle U, V \rangle\, |\mathbf{x}_u \times \mathbf{x}_v|\, du\, dv$$

$$\therefore \quad \boxed{A'(0) = -\iint_M 2H \langle U, V \rangle \, dA}$$

where dA = area element of M.

(This is the First Variation formula)

Note: More generally, we have (Ex 4.4.2 of Oprea)

$$\boxed{A'(0) = -\iint_M 2H \langle U, V \rangle \, dA - \int_{\partial M} \langle V, U \times T \rangle \, ds}$$

where ∂M = boundary of M

T = unit tangent of ∂M

ds = arc-length element of ∂M

Application of 1^{st} variational formula

(i) Take the special case that $V = X(u,v)$.

Then $Y^t = (1+t)\,X$

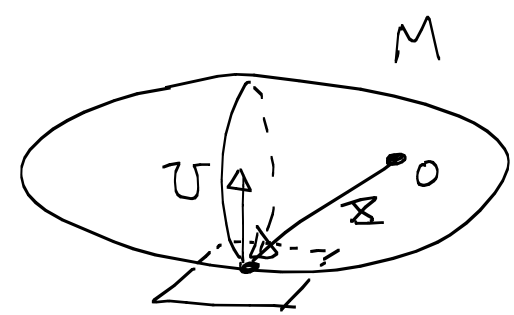
$\therefore$ M^t = scaling of M by a factor $1+t$

$\Rightarrow$ $A(t) = (1+t)^2 A$, where $A = A(0)$ = area of M.

1^{st} variational formula $\Rightarrow$

$$2A = A'(0) = -\iint_M 2H\langle U, X\rangle \, dA$$

$\therefore$ $\boxed{A = -\iint_M H\langle U, X\rangle \, dA}$ for surface in $\mathbb{R}^3$



(ii) Take the special case that $V = f\nu$, f = function on M.

Then 1^{st} variational formula $\Rightarrow$

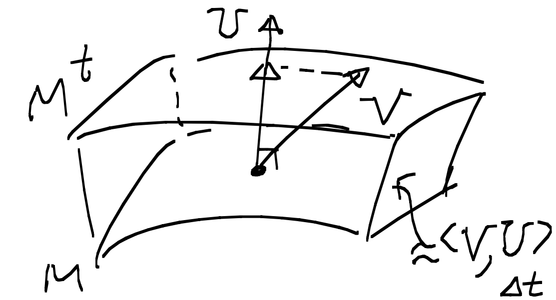
$$A'(0) = -\iint_M 2H \langle \nu, f\nu \rangle dA$$

$$\Rightarrow \quad \boxed{A'(0) = -2\iint_M fH\, dA}$$

(↳ integration of the function fH on M)

Now suppose M encloses a volume Ω.

Then for t sufficiently small, M^t also encloses a volume Ω_t. By advanced calculus



$$\left.\frac{d}{dt}\right|_{t=0} Vol(\Omega_t) = \iint_{\partial\Omega = M} \langle V, \nu \rangle dA$$

$$= \iint_M \langle f U, U \rangle \, dA \qquad \text{(in our case)}$$

$$= \iint_M f \, dA .$$

$\therefore$ If all Ω_t have the same volume, then

$$\boxed{\iint_M f \, dA = 0}$$

Therefore, if M has the least surface area among all (nearby) surfaces enclosing the same volume, then the mean curvature H of M satisfies

$$\boxed{\iint_M f H \, dA = 0 \quad \forall \text{ function } f \text{ with } \iint_M f \, dA = 0} .$$

Ex 4.4.3 of Oprea $\Rightarrow$ $\boxed{H \equiv \text{const.}}$

$\left(\begin{array}{l} \text{Calculate} \\ \displaystyle\iint_M (H-c)^2 \, dA, \\ \text{where} \quad \displaystyle c = \frac{1}{A(M)} \iint_M H \, dA. \end{array} \right)$

$\therefore$ we've the following

Thm (Thm 4.4.4 of Oprea)

A soap bubble must always take the form of a surface of constant mean curvature.

However, to see the soap bubble more geometrically, we have

Thm (Alexandrov, Thm 4.4.6 of Oprea)

If M is a compact embedded (i.e. no self intersection) surface of constant mean curvature, then M is a standard sphere.

($\therefore$ stable soap bubbles are standard spheres)

To prove the Alexandrov Theorem, we need

Thm (Ros , Thm 4.4.5 of the text book)

- Let M be a compact embedded surface in $\mathbb{R}^3$ bounding a domain D of volume Vol. If $H > 0$ on M, then

$$\iint_M \frac{1}{H}\, dA \geq 3\,\mathrm{Vol}$$

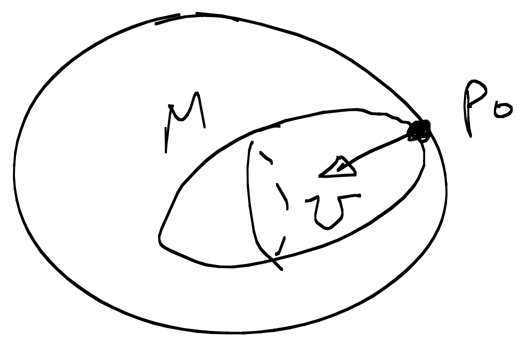
- Furthermore, equality holds $\Leftrightarrow$ M is a standard sphere.

Pf of Alexandrov Theorem

Note that M cpt $\Rightarrow$ $\exists\, p_0 \in M$ s.t

$K > 0$, i.e $k_1(p_0)$, $k_2(p_0)$ are of same sign.

Recall that p_0 is the tangent point of M with a sphere:



Therefore, choosing ν = inward normal makes

$$k_1(p_0),\ k_2(p_0) > 0$$

$$\Longrightarrow \quad H(p_0) > 0$$

$$\Longrightarrow \quad H > 0. \quad (\text{since } H \equiv \text{const.})$$

Now, application (i) $\Rightarrow$

$$A = -\iint_M H \langle X, \nu \rangle \, dA$$

$$= -H \iint_M \langle X, \nu \rangle \, dA. \qquad (H = \text{const.})$$

By divergence theorem,

$$A = H \iiint_D \operatorname{div}(X) \, d\mathrm{vol} \qquad \left(\begin{array}{c}\text{since} \\ \nu = \text{inward unit} \\ \text{normal}\end{array}\right)$$

$$= H \iiint_D \left(\frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} + \frac{\partial x_3}{\partial x_3} \right) d\mathrm{vol} \qquad (\partial D = M)$$

$$= 3H \, \mathrm{Vol}.$$

$$\Longrightarrow \quad \iint_M \frac{1}{H} \, d\mathrm{vol} = \frac{1}{H} \iint_M dA = \frac{A}{H} = 3 \, \mathrm{Vol}.$$

Hence Ros' thm $\Longrightarrow$ M = standard sphere.

#

Proof of Ros' Theorem

<u>Step 1</u>: The domain D bounded by M can be parametrized locally by

$$Y(u,v,t) = X(u,v) + t\,U(u,v)$$

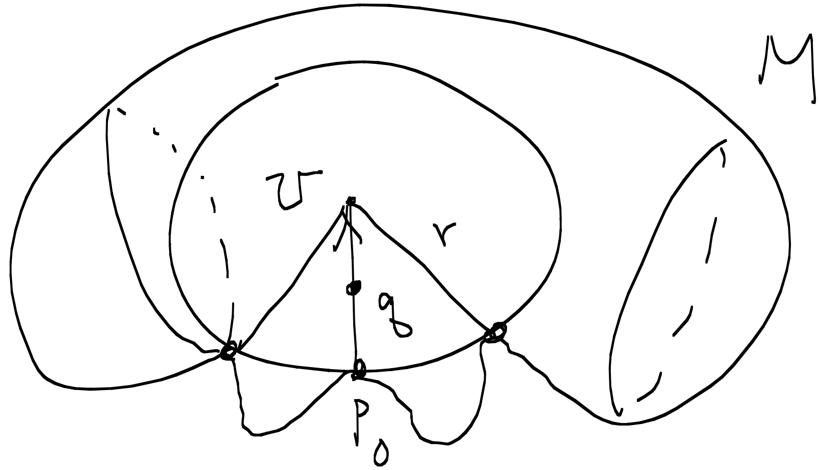
where $X(u,v) =$ local parametrization of M,

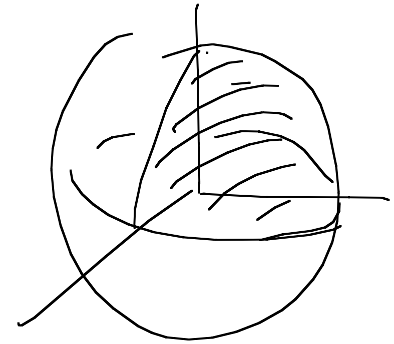
$$U(u,v) = \frac{X_1 \times X_2}{|X_1 \times X_2|} = \underline{\text{inward}} \text{ unit normal},$$

and $t \in [0, h(u,v)]$,

with $h(u,v) = h(p_0)$

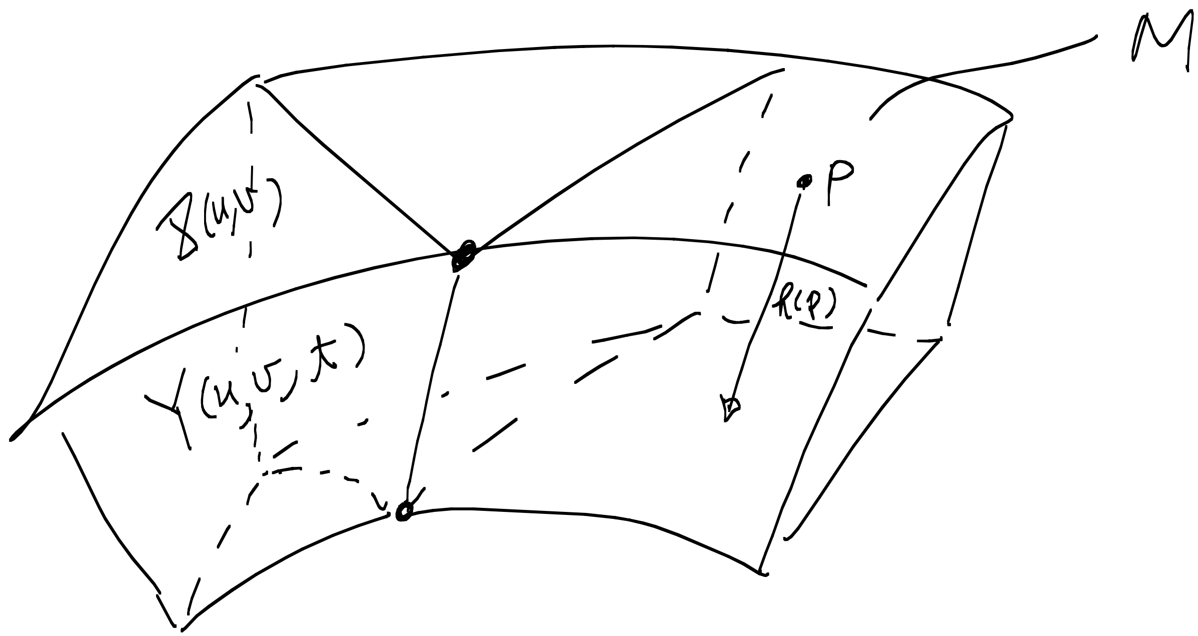
$$= \sup\left\{ r : \begin{array}{l} \forall\, q = p_0 + \tau U,\ 0 < \tau < r, \\ p_0 = \text{unique minimum of } |q - p| \text{ for } p \in M \end{array} \right\}$$





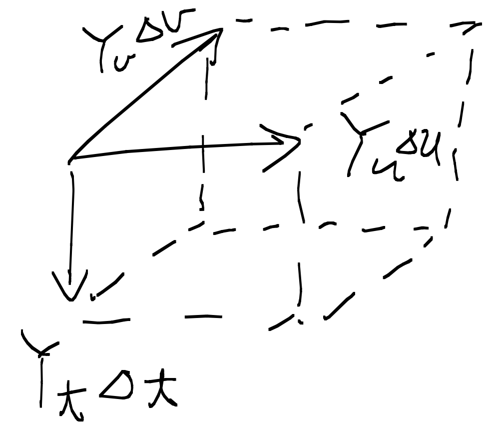
Step 2 Volume of D in terms of $\{Y(u,v,t)\}$

Note that if we choose $\{X(u,v)\}$ so that the coordinate patches have no interior intersection, then the corresponding $\{Y(u,v,t)\}$ also have no interior intersection.



Therefore, we can calculate the volume of D by summing up the volumes of each local coordinates $Y(u,v,t)$.

Recall that the volume of the approximating parallelepiped determined by $\quad Y_u \Delta u, \ Y_v \Delta v \ \& \ Y_t \Delta t$



is given by

$$\left| \langle Y_u \Delta u \times Y_v \Delta v, \ Y_t \Delta t \rangle \right|$$

$$\simeq \left| \langle Y_u \times Y_v, \ Y_t \rangle \right| \, du \, dv \, dt$$

Hence
$$\mathrm{Vol}(D) = \iint_M \left[\int_0^{R(u,v)} \left| \langle Y_u \times Y_v, \ Y_t \rangle \right| dt \right] du \, dv$$

By $\quad Y = \gamma + t V, \quad$ we have

$$\begin{cases} Y_u = X_u + t\,U_u \\ Y_v = X_v + t\,U_v \\ Y_t = U \end{cases}$$

We may assume $X(u,v)$ is chosen so that $X(u, v_0)$ & $X(u_0, v)$ are line of curvatures for the moment.

Then $$\begin{cases} U_u = -S(X_u) = -k_1 X_u \\ U_v = -S(X_v) = -k_2 X_v \end{cases}$$

$$\Rightarrow \begin{cases} Y_u = (1 - k_1 t) X_u \\ Y_v = (1 - k_2 t) X_v \end{cases}$$

$$\Rightarrow \quad Y_u \times Y_v = (1-k_1 t)(1-k_2 t)\; X_u \times X_v$$

$$= (1 - 2Ht + Kt^2)\; X_u \times X_v$$

$$\Rightarrow \quad |\langle Y_u \times Y_v, Y_t \rangle| = |1-2Ht+Kt^2|\, |\langle X_u \times X_v, U \rangle|$$

$$= |1-2Ht+Kt^2|\; |X_u \times X_v|$$

$$\therefore \quad Vol(D) = \iint_M \left[\int_0^{h(u,v)} |1-2Ht+Kt^2|\; \underbrace{|X_u \times X_v|}_{\text{independent of } t}\, dt \right] du\,dv$$

$$= \iint_M \left[\int_0^{h(u,v)} |1-2Ht+Kt^2|\, dt \right] |X_u \times X_v|\; du\,dv$$

$$\Rightarrow \quad \boxed{Vol(D) = \iint_M \left[\int_0^{h(u,v)} |1 - 2Ht + Kt^2| \, dt \right] dA}$$

Note that mean curvature H, Gauss curvature K, & dA are independent of coordinates!

We may choose <u>any coordinates</u> $X(u,v)$ to calculate $Vol(D)$ by the above formula. (not necessary with lines of curvature)

<u>Step3</u> Estimate of normal curvature by the function $h(p) = h(u,v)$ $\quad (p = X(u,v))$:

$$\frac{1}{h(p)} \geq \max\{k_1(p), k_2(p)\}, \quad \forall p \in M.$$

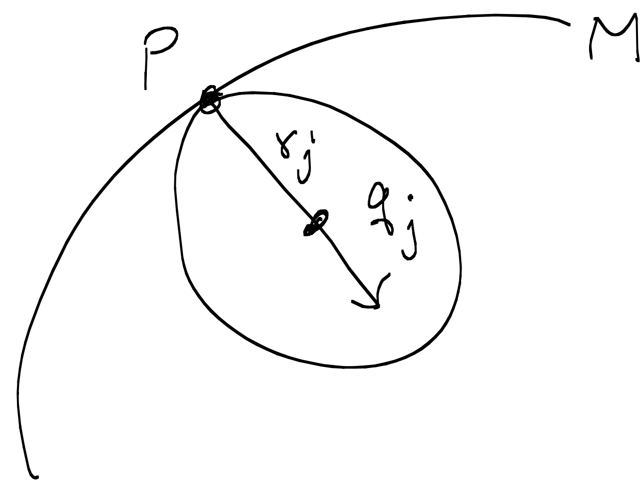
In fact, $\forall p \in M$ the definition of $h(p) \Rightarrow \exists$ sequence

of radius $r_j \longrightarrow h(p)$ s.t.

$p =$ unique minimum of the distance

from $q_j = p + r_j \, \nu(p)$ to the

surface M



This implies the

sphere

$S^2(q_j; r_j)$ ← center ← radius

touches M at p and completely contained in D.

Use the method of showing cpt embedded surface in $\mathbb{R}^3$ has positive K at some point, we see that

$$\max\{k_1(p), k_2(p)\} \leq \text{normal curvature of } S^2(q_j; r_j) = \frac{1}{r_j}.$$

Letting $j \to \infty$, we have

$$\max\{k_1(p), k_2(p)\} \leq \frac{1}{h(p)}, \quad \forall\, p \in M.$$

<u>Step 4</u>: $3\,\mathrm{Vol}(D) \leq \iint_M \frac{1}{H}\, dA.$

By step 3, $0 < H(p) = \dfrac{k_1(p) + k_2(p)}{2} \leq \dfrac{1}{h(p)}$

$$\Rightarrow \quad 1 - 2Ht + Kt^2 = (1 - k_1 t)(1 - k_2 t)$$

$$\geq \left(1 - \frac{1}{h} t\right)^2 \quad \left(\text{since } \begin{array}{c} 0 \leq t \leq h \\ k_i t \leq \frac{1}{h} t \leq 1 \end{array}\right)$$

$$\geq 0$$

$$\Rightarrow \quad \int_0^{h(p)} |1 - 2Ht + Kt^2| \, dt = \int_0^{h(p)} (1 - 2Ht + Kt^2) \, dt$$

$$= \int_0^{h(p)} \left[(1 - Ht)^2 - (H^2 - K)t^2\right] dt$$

$$H^2 - K = \left(\frac{k_1+k_2}{2}\right)^2 - k_1k_2 = \left(\frac{k_1-k_2}{2}\right)^2 \geq 0$$

$$\Rightarrow \quad \int_0^{h(p)} \left|1-2Ht+Kt^2\right| dt \leq \int_0^{h(p)} (1-Ht)^2\, dt$$

$$\left(\text{since } H \leq \frac{1}{h(p)}\right) \qquad \leq \int_0^{\frac{1}{H}} (1-Ht)^2\, dt$$

$$= \left[t - Ht^2 + \frac{H^2t^3}{3}\right]_0^{\frac{1}{H}}$$

$$= \frac{1}{H} - \frac{1}{H} + \frac{1}{3H} = \frac{1}{3H}$$

Therefore
$$\mathrm{Vol}(D) = \iint_M \left[\int_0^{h(p)} |1 - 2Ht + Kt^2| \, dt \right] dA$$
$$\leq \iint_M \frac{1}{3H} \, dA = \frac{1}{3} \iint_M \frac{1}{H} \, dA .$$

Final Step :

If equality holds, then all inequalities above have to be equalities. In particular

$$H^2 - K = 0$$
$$\Rightarrow \quad k_1 = k_2 \quad \Rightarrow \quad \text{totally umbilic}$$
$$\overset{(H>0)}{\Longrightarrow} \quad M = \text{standard sphere} .$$

(§ 4.5 – 4.9 Omitted)

Ch 5 Geodesics

Let α be a curve on a surface $M \subset \mathbb{R}^3$.

As a space curve

$$\alpha''(s) = \kappa_\alpha N(s), \quad s = \text{arc length}$$

where κ_α = curvature of α

$N(s) = T'(s)$ is the normal of α

Note that $\alpha \in M$

$$\Rightarrow \quad T = \alpha' \in T_\alpha M$$

But $N = T' \neq U$ surface normal in general

(Of course, U is contained in the $\{N, B\}$ plane)

Since $\alpha \in M$, the surface normal U is a distinguish normal to α. One should use U instead of N in order to reflect the geometry of M.

Now $\{T, U \times T, U\}$ $\left(T, U\times T \in T_\alpha M \atop U \text{ normal to } T_\alpha M \right)$

forms a orthonormal basis of $\mathbb{R}^3$ and hence

$$\alpha'' = AT + B\, U\times T + C\,U$$

for some scalars A, B, C.

- s = arc-length $\Rightarrow \langle \alpha', \alpha' \rangle = 1$

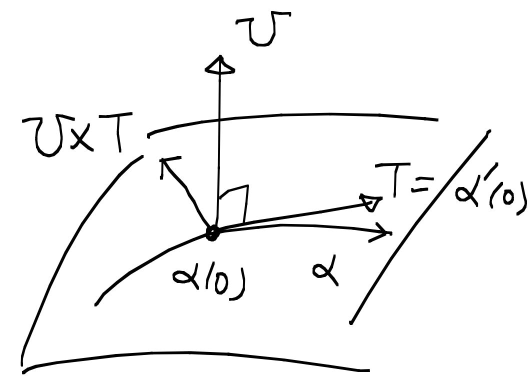
$$\Rightarrow \quad \langle \alpha'', \alpha' \rangle = 0$$

$$\Rightarrow \quad A = \langle \alpha'', T \rangle = \langle \alpha'', \alpha' \rangle = 0.$$

- On the other hand,

$$C = \langle \alpha'', U \rangle = \langle S(\alpha'), \alpha' \rangle = k(\alpha')$$

is the normal curvature of M in α'-direction



Together we have

$$\alpha'' = \underbrace{B\, U \times T}_{\text{tangential}} + \underbrace{k(\alpha')\, U}_{\text{normal}}$$

<u>Def</u> : The <u>geodesic curvature</u> of α is defined by

$$\boxed{K_g = \langle \alpha'', U \times T \rangle}.$$

Therefore

$$\boxed{\alpha'' = K_g\, U \times T + k(\alpha')\, U.}$$

Note that $U \times T \in T_{\alpha(s)}M$

$$\begin{cases} \kappa_g\, U \times T = \text{tangential part of } \alpha'' = (\alpha'')^T = (\alpha'')_{\tan} \\ k(\alpha')\, U = \text{normal part of } \alpha'' = (\alpha'')^\perp = (\alpha'')_{\text{normal}} \end{cases}$$

<u>Physically</u>, α'' = acceleration (in $\mathbb{R}^3$),

$\therefore$ $\kappa_g\, U \times T$ = <u>acceleration seen in M</u>

i.e. κ_g = curvature of α with respect to the internal geometry of M.

<u>Remarks</u>: (1) $\kappa_g = \langle \alpha'', U \times T \rangle$

$$= \langle \alpha'', U \times \alpha' \rangle$$

$$= \langle U, \alpha' \times \alpha'' \rangle$$

$$= \langle U, \kappa_\alpha T \times N \rangle$$

$$\therefore \quad \boxed{\kappa_g = \kappa_\alpha \cos\theta}$$

where θ = angle between U & $T \times N = B$.

(surface normal) (binormal of α.).

κ_α = curvature of α in $\mathbb{R}^3$

(2) It is clear from

$$\alpha'' = \kappa_g\, U \times T + k(\alpha')\, U$$

$$\left(\begin{array}{c} \| \\ \kappa_\alpha N \end{array} \right)$$

that $$\boxed{\kappa_\alpha^2 = \left(k(\alpha')\right)^2 + \kappa_g^2}$$ (Ex. 5.1.1)

Formula for geodesic curvature κ_g

From
$$\alpha'' = \kappa_g\, U \times T + k(\alpha')\, U$$
we need to calculate the tangential component of α''.

Let $\mathbf{x}(u,v)$ be a coordinate patch of M, then α is represented as $\alpha(s) = \mathbf{x}(u(s), v(s))$, $s =$ arc-length.

$$\Rightarrow \quad T = \alpha' = \sum_{i=1}^{2} \mathbf{x}_i \frac{du^i}{ds}$$

$$\& \quad \alpha'' = \sum_{i=1}^{2} \mathbf{x}_i \frac{d^2u^i}{ds^2} + \sum_{i=1}^{2}\left(\sum_{j=1}^{2} \mathbf{x}_{ij} \frac{du^j}{ds}\right)\frac{du^i}{ds}$$
$$= \sum_{i=1}^{2} \mathbf{x}_i \frac{d^2u^i}{ds^2} + \sum_{i,j=1}^{2} \mathbf{x}_{ij} \frac{du^i}{ds}\frac{du^j}{ds}$$

Recall $$X_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^{k} X_k + h_{ij} U$$

where $\{\Gamma_{ij}^{k}\}$ = Christoffel symbols , $h_{ij} = \langle X_{ij}, U \rangle$

$$\Rightarrow \quad \alpha'' = \sum_{\cancel{i}\,k=1}^{2} X_{\cancel{i}\,k} \frac{d^2 u^{\cancel{i}\,k}}{ds^2} + \sum_{i,j=1}^{2} \left(\sum_{k=1}^{2} \Gamma_{ij}^{k} X_k \right) \frac{du^i}{ds} \frac{du^j}{ds}$$

$$+ \left(\sum_{i,j=1}^{2} h_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \right) U$$

$$= \sum_{k=1}^{2} \left(\frac{d^2 u^k}{ds^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^{k} \frac{du^i}{ds} \frac{du^j}{ds} \right) X_k$$

$$+ \left(\sum_{i,j=1}^{2} h_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \right) U$$

(tangential part of α'')

$$\therefore \quad \boxed{\alpha''_{tan} = \sum_{k=1}^{2} \left(\frac{d^2 u^k}{ds^2} + \sum_{i,j=1}^{2} \Gamma_{ij}^{k} \frac{du^i}{ds} \frac{du^j}{ds} \right) X_k}$$

i.e. $$\boxed{\kappa_g\, U\times T = \sum_{k=1}^{2}\left(\frac{d^2u^k}{ds^2}+\sum_{i,j=1}^{2}\Gamma_{ij}^{k}\frac{du^i}{ds}\frac{du^j}{ds}\right)X_k}$$

Let $$\frac{D}{ds}\left(\frac{du^k}{ds}\right)=\frac{d^2u^k}{ds^2}+\sum_{i,j=1}^{2}\Gamma_{ij}^{k}\frac{du^i}{ds}\frac{du^j}{ds},$$

then $$\kappa_g\, U\times T = \sum_{k=1}^{2}\left(\frac{D}{ds}\left(\frac{du^k}{ds}\right)\right)X_k$$

Note that $T=\sum_k \frac{du^k}{ds}X_k=\frac{du^1}{ds}X_1+\frac{du^2}{ds}X_2$,

we have
$$\begin{aligned}\kappa_g U &= \kappa_g\, T\times(U\times T)\\ &= T\times(\kappa_g U\times T)\\ &= \left(\frac{du^1}{ds}X_1+\frac{du^2}{ds}X_2\right)\times\left(\frac{D}{ds}\left(\frac{du^1}{ds}\right)X_1+\frac{D}{ds}\left(\frac{du^2}{ds}\right)X_2\right)\end{aligned}$$

$$= \left[\frac{du^1}{ds}\frac{D}{ds}\left(\frac{du^2}{ds}\right) - \frac{du^2}{ds}\frac{D}{ds}\left(\frac{du^1}{ds}\right)\right] \mathbf{X}_1 \times \mathbf{X}_2$$

$$= \left[\frac{du^1}{ds}\frac{D}{ds}\left(\frac{du^2}{ds}\right) - \frac{du^2}{ds}\frac{D}{ds}\left(\frac{du^1}{ds}\right)\right] |\mathbf{X}_1 \times \mathbf{X}_2|\, \mathbf{U}$$

$$= \left[\frac{du^1}{ds}\frac{D}{ds}\left(\frac{du^2}{ds}\right) - \frac{du^2}{ds}\frac{D}{ds}\left(\frac{du^1}{ds}\right)\right] \sqrt{\det(g_{ij})}\, \mathbf{U}$$

This shows that

$$\boxed{\begin{aligned} K_g &= \sqrt{\det(g_{ij})}\left(\frac{du^1}{ds}\frac{D}{ds}\left(\frac{du^2}{ds}\right) - \frac{du^2}{ds}\frac{D}{ds}\left(\frac{du^1}{ds}\right)\right) \\ \text{where } \frac{D}{ds}\left(\frac{du^k}{ds}\right) &= \frac{d^2u^k}{ds^2} + \sum_{i,j=1}^{2} \Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds}. \end{aligned}}$$

Thm The geodesic curvature κ_g depends only on α & the metric coefficients.

Pf By the about formula and

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(\frac{\partial g_{lj}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\right)$$

where $(g^{kl}) = (g_{ij})^{-1}$. #

Special case $(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix} \Rightarrow \det(g_{ij}) = EG$

Then

$$\begin{cases} \Gamma_{11}^{1} = \frac{1}{2E}E_1, & \Gamma_{11}^{2} = -\frac{1}{2G}E_2 \\ \Gamma_{12}^{1} = \frac{1}{2E}E_2, & \Gamma_{12}^{2} = \frac{1}{2G}G_1 \\ \Gamma_{22}^{1} = -\frac{1}{2E}G_1, & \Gamma_{22}^{2} = \frac{1}{2G}G_2 \end{cases}$$

$$\Rightarrow \quad \frac{D}{ds}\left(\frac{du^1}{ds}\right) = \frac{d^2u^1}{ds^2} + \Gamma^1_{11}\left(\frac{du^1}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^1_{22}\left(\frac{du^2}{ds}\right)^2$$

$$= \frac{d^2u^1}{ds^2} + \frac{E_1}{2E}\left(\frac{du^1}{ds}\right)^2 + \frac{E_2}{E}\frac{du^1}{ds}\frac{du^2}{ds} - \frac{G_1}{2E}\left(\frac{du^2}{ds}\right)^2$$

$$\frac{D}{ds}\left(\frac{du^2}{ds}\right) = \frac{d^2u^2}{ds^2} + \Gamma^2_{11}\left(\frac{du^1}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^2_{22}\left(\frac{du^2}{ds}\right)^2$$

$$= \frac{d^2u^2}{ds^2} - \frac{E_2}{2G}\left(\frac{du^1}{ds}\right)^2 + \frac{G_1}{G}\frac{du^1}{ds}\frac{du^2}{ds} + \frac{G_2}{2G}\left(\frac{du^2}{ds}\right)^2$$

$$\Rightarrow \quad k_g = \sqrt{EG}\left\{ \begin{array}{l} u'v'' - \frac{E_v}{2G}(u')^3 + \frac{G_u}{G}(u')^2 v' + \frac{G_v}{2G}u'(v')^2 \\ - \left[v'u'' + \frac{E_u}{2E}(u')^2 v' + \frac{E_v}{E}u'(v')^2 - \frac{G_u}{2E}(v')^3 \right] \end{array} \right\}$$

$\therefore$ We have

<u>Thm</u> (Thm 5.1.5 of Oprea) If $(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$, then

$$\boxed{\begin{aligned} K_g = \sqrt{EG}\Big\{ (u'v'' - v'u'') - \frac{E_v}{2G} u'^3 + \left(\frac{G_u}{G} - \frac{E_u}{2E}\right) u'^2 v' \\ + \left(\frac{G_v}{2G} - \frac{E_v}{E}\right) u' v'^2 + \frac{G_u}{2E} v'^3 \Big\} \end{aligned}}$$

where " ' " denote derivatives with respect to a arc-length parameter.

Remark: Suppose $\alpha(t)$ with general parameters.

Then $$\frac{d\alpha}{ds} = \alpha'(t)\frac{dt}{ds} \qquad (\cdot)' = \text{derivative wrt } t$$

$$\Rightarrow \quad \frac{d^2\alpha}{ds^2} = \alpha''(t)\left(\frac{dt}{ds}\right)^2 + \alpha'(t)\frac{d^2t}{ds^2}$$

$$\Rightarrow \quad \left\langle \frac{d^2\alpha}{ds^2}, U\times T\right\rangle = \left(\frac{dt}{ds}\right)^2 \langle \alpha''(t), U\times T\rangle$$

$$\left(\text{since} \quad \langle \alpha', U\times T\rangle = |\alpha'|\underbrace{\langle T, U\times T\rangle}_{=0} \right)$$

Since $$1 = \left|\frac{d\alpha}{ds}\right| = |\alpha'(t)|\left|\frac{dt}{ds}\right|,$$

$$\boxed{\kappa_g = \frac{1}{|\alpha'(t)|^2}\langle \alpha''(t), U\times T\rangle} \quad \text{——} \quad (*)$$

$$= \frac{1}{|\alpha'|^2} \langle \alpha'', U \times \frac{\alpha'}{|\alpha'|} \rangle$$

$$\therefore \quad \boxed{\kappa_g = \frac{1}{|\alpha'|^3} \langle U, \alpha' \times \alpha'' \rangle}$$

From (*), $\kappa_g = 0 \Rightarrow \alpha''(t)$ has no $U \times T$ component.

However, what we need is

__Def.__ A curve α on M is a __geodesic__ if

$$\alpha''_{tan} = 0. \qquad \text{(general parameters)}$$

__Lemma__ (Lemma 5.1.7 of Oprea)

If α is a geodesic, then $|\alpha'| = \text{constant}$.

Pf:

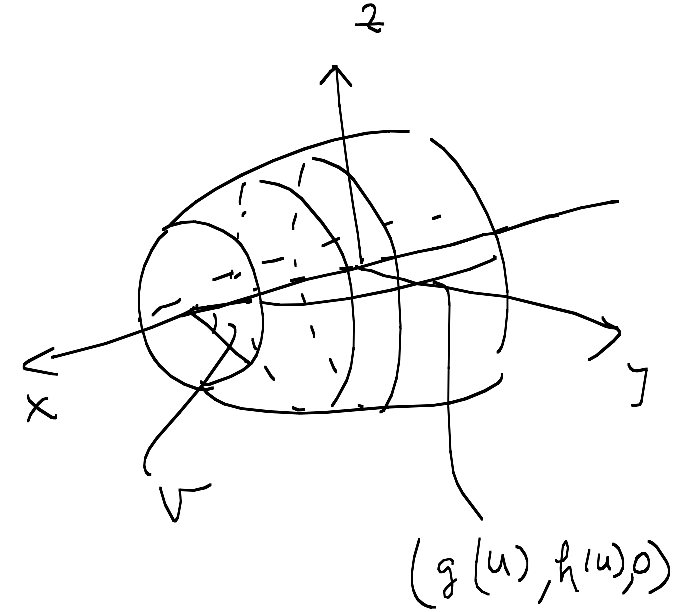
$$\frac{d}{dt}|\alpha'|^2 = \frac{d}{dt}\langle \alpha', \alpha' \rangle = 2\langle \alpha', \alpha'' \rangle$$

$$= 2\langle \alpha', \alpha''_{tan} + \alpha''_{normal} \rangle$$

$$= 2\langle \alpha', \alpha''_{tan} \rangle \quad \text{since } \alpha' \in T_\alpha M$$

$$= 0 \quad \text{since } \alpha \text{ is a geodesic.} \quad \#$$

eg: Consider surface of revolution (about x-axis):



$$X(u,v) = (g(u), h(u)\cos v, h(u)\sin v)$$

Recall

$$\begin{cases} X_1 = (g', h'\cos v, h'\sin v) \\ X_2 = (0, -h\sin v, h\cos v) \end{cases}$$

$$X_{11} = (g'', h''\cos v, h''\sin v)$$

$$\Rightarrow \langle X_{11}, X_1 \rangle = g'g'' + h'h'' = \frac{1}{2}\left(g'^2 + h'^2\right)'$$

$$\langle X_{11}, X_2 \rangle = 0$$

$$\therefore \quad (X_{11})_{tan} = 0 \iff (g'^2 + h'^2)' = 0$$

In particular, if the meridian $\alpha(u) = (g(u), h(u), 0)$ is parametrized by arc length, i.e $|\alpha'(u)| = 1$.

$$\Longrightarrow \quad g'^2 + h'^2 \equiv 1 \Rightarrow (g'^2 + h'^2)' = 0$$

$$\Longrightarrow \quad (X_{11})_{tan} = 0$$

$\therefore$ If the meridians are parametrized by arc-length, then they are geodesic.

eg: Geodesics on S_R^2 = sphere of radius R.

S_R^2 = surface of revolution parametrized by

$$X(u,v) = (R\cos u \cos v,\ R\sin u \cos v,\ R\sin v)$$

(Note: z-axis = axis of revolution,

meridian $= (0,\ R\cos v,\ R\sin v)$)

i.e. $g(v) = R\sin v, \quad h(v) = R\cos v$

$\Rightarrow \quad g'^2 + h'^2 = R^2\cos^2 v + R^2\sin^2 v = R^2$ constant

$\Rightarrow$ $\quad \mathbf{X}(u_0, v) = $ geodesic $\quad \forall$ fixed u_0

i.e. longitudes from North to South pole

are geodesics.

Symmetry $\Rightarrow$ great circles are geodesics

(Recall : great circles $= S_R^2 \cap$ Plane passing thro center of S_R^2)

Conversely, if $\alpha(s)$ is a geodesic on S_R^2

parametrized by arc-length. Then

$$\alpha'' = \alpha''_{tan} + \alpha''_{normal}$$

$$= \alpha''_{tan} + \langle \alpha'', U \rangle U$$

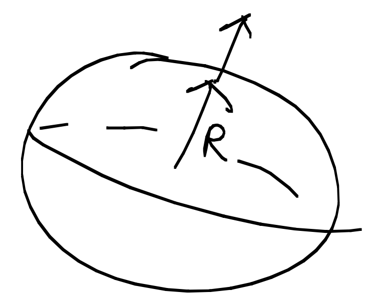
α is a geodesic $\Longrightarrow$

$$\alpha'' = \langle \alpha'', U \rangle U$$

Note that the outward normal is

$$U = \frac{\alpha(t)}{R} \quad \text{at } \alpha(t)$$

$$\Longrightarrow \quad \alpha'' = \langle \alpha'', \alpha \rangle \frac{\alpha}{R^2}$$



On the other hand

$$(U \times T)' = \left(\frac{\alpha \times \alpha'}{R} \right)'$$

$$= \frac{\alpha' \times \alpha' + \alpha \times \alpha''}{R}$$

$$= \frac{\alpha \times \frac{\langle \alpha'', \alpha \rangle}{R^2} \alpha}{R} = 0$$

$\Rightarrow \tilde{N} = U \times T$ is a constant tangent vector s.t.

$$\langle \tilde{N}, \alpha \rangle = \langle U \times T, \alpha \rangle = \langle \frac{\alpha \times \alpha'}{R}, \alpha \rangle = 0$$

$\Rightarrow$ α belongs to the plane with normal $\tilde{N}$

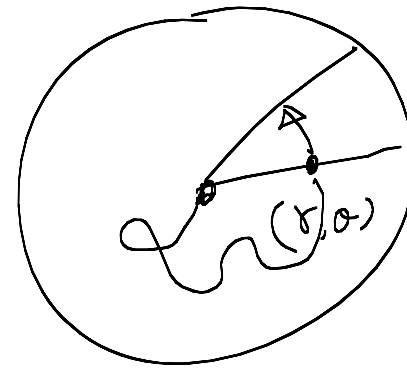
Moreover, $\langle U, \tilde{N} \rangle = 0$

$\Rightarrow$ U belongs to the plane

$\Rightarrow$ the plane contains the center of S^2_R

$\Rightarrow$ α = great circle. #

Geodesic Polar coordinates:



Euclidean: $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$\mathbf{x}(u,v)$ with

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}$$

i.e. $|\mathbf{x}_u|^2 = 1$, $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$

and $|\mathbf{x}_v|^2 = G(u,v) > 0$

(e.g. surface of revolution generated by unit speed curve)

Then $\alpha(s) = \mathbf{x}(s, v_0)$ is a geodesic

parametrized by arc-length $s \in [0, s_1]$:

$$\begin{cases} \alpha'(s) = X_1 \\ \alpha''(s) = X_{11} \end{cases} \qquad \left(\begin{array}{c} s_1 = L(\alpha) \\ \Uparrow \end{array} \right)$$

$$\Rightarrow \text{①} \quad |\alpha'(s)|^2 = |X_1|^2 = 1 \Rightarrow s = \text{arc-length}$$

$$\text{②} \quad \langle \alpha'', X_1 \rangle = \langle X_{11}, X_1 \rangle = \tfrac{1}{2} \langle X_1, X_1 \rangle_1 = 0$$

$$\langle \alpha'', X_2 \rangle = \langle X_{11}, X_2 \rangle = \cancelto{0}{\langle X_1, X_2 \rangle_1} - \langle X_1, X_{21} \rangle$$

$$= -\tfrac{1}{2} \langle X_1, X_1 \rangle_2 = 0$$

$$\therefore \quad \alpha''_{tan} = 0 \Rightarrow \alpha \text{ is a geodesic.}$$

Now suppose we have another curve $\beta(s)$, $s \in [0, s_1]$

such that

$$\beta(0) = \alpha(0) = \mathbf{X}(0, v_0), \quad \beta(s_1) = \alpha(s_1) = \mathbf{X}(s_1, v_0)$$

i.e. $\quad u(0) = 0, \quad u(s_1) = s_1.$

Let $\beta(s) = \mathbf{X}(u(s), v(s))$

Then $\quad \beta'(s) = \mathbf{X}_1 u' + \mathbf{X}_2 v'$

$$\Rightarrow \quad L(\beta) = \int_0^{s_1} \sqrt{\langle \beta'(s), \beta'(s) \rangle}\, ds$$

$$= \int_0^{s_1} \sqrt{u'^2 + G v'^2}\, ds$$

$$\geq \int_0^{s_1} |u'|\, ds \geq \int_0^{s_1} u'\, ds$$

$$= u(s_1) - u(0) = s_1 = L(\alpha)$$

$\Rightarrow \quad \alpha$ is length minimizing.

Note that $L(\beta) = L(\alpha) \iff$

$$u' > 0 \quad \& \quad v' \equiv 0$$

$$\Rightarrow \quad \beta(s) = X(u(s), v_0)$$

$\Rightarrow \quad \beta \equiv \alpha$ up to reparametrization.

5.2 Geodesic Equations and the Clairaut Relation

Let $\mathbf{X}(u,v)$ be a coordinate patch of a regular surface M. Then $\forall$ curve $\alpha(t) = \mathbf{X}(u(t), v(t))$ on M, we have

$$\alpha''_{\tan} = \sum_{k=1}^{2} \left(\frac{d^2 u^k}{dt^2} + \sum_{i,j=1}^{2} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right) \mathbf{X}_k$$

(check!)

By definition of geodesic,

α is a geodesic $\iff$ $\alpha''_{\tan} = 0$

$\iff$ $\alpha(t) = \mathbf{X}(u(t), v(t))$ satisfies the Geodesic Equations:

$$\boxed{\frac{d^2 u^k}{dt^2} + \sum_{i,j=1}^{2} \Gamma^k_{ij} \frac{du^i}{dt} \frac{du^j}{dt} = 0, \quad k = 1, 2}$$

In (u,v) notation,

$$\boxed{\begin{cases} u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2 = 0 \\ v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2 = 0 \end{cases}}$$

<u>Special case</u>:

If $\mathbf{x}(u,v)$ satisfies $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$, then in classical notation $(g_{ij}) = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$ and

$$\begin{cases} \Gamma^1_{11} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1} \right) = \frac{E_u}{2E} \\ \Gamma^1_{12} = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^1} \right) = \frac{E_v}{2E} \\ \Gamma^1_{22} = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1} \right) = -\frac{G_u}{2E} \end{cases}$$

$$\left\{\begin{aligned}
\Gamma^2_{11} &= \frac{1}{2} g^{22}\left(\frac{\partial g_{21}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right) = -\frac{E_v}{2G} \\
\Gamma^2_{12} &= \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^2}\right) = \frac{G_u}{2G} \\
\Gamma^2_{22} &= \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2}\right) = \frac{G_v}{2G}
\end{aligned}\right.$$

And the geodesic equations become

$$\left\{\begin{aligned}
& u'' + \frac{E_u}{2E} u'^2 + \frac{E_v}{E} u'v' - \frac{G_u}{2E} v'^2 = 0 \\
& v'' - \frac{E_v}{2G} u'^2 + \frac{G_u}{G} u'v' + \frac{G_v}{2G} v'^2 = 0
\end{aligned}\right.$$

The following is a theorem from the theory of ODE :

<u>Thm</u> (Thm 5.2.3 of Oprea)

Let $p = \mathbf{x}(u_0, v_0)$ be a point on a surface $M : \mathbf{x}(u,v)$ and $V \in T_pM$ be a tangent vector to M at p.

Then there exists a unique geodesic $\alpha : (-\varepsilon, \varepsilon) \to M$ defined on some open interval $(-\varepsilon, \varepsilon)$ such that

$$\begin{cases} \alpha(0) = p \quad \& \\ \alpha'(0) = V \end{cases}$$

(Pf: Initial Value Problem for ODEs.)

Remark: Whether the interval of definition $(-\varepsilon, \varepsilon)$ of the geodesic α can be extended to $(-\infty, \infty)$ is an important issue. If it is so $\forall p$ & $\forall V$, then the surface M is said to be complete. (Next section!)

eg <u>Unit Sphere S^2</u> : use standard patch

$$\mathbf{x}(u,v) = (\cos u \cos v,\ \sin u \cos v,\ \sin v)$$

we have

$$\mathbf{x}_1 = (-\sin u \cos v,\ \cos u \cos v,\ 0)$$

$$\mathbf{x}_2 = (-\cos u \sin v,\ -\sin u \sin v,\ \cos v)$$

$$\Rightarrow \quad (g_{ij}) = \begin{pmatrix} \cos^2 v & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{note: indep. of } u)$$

$$\Rightarrow \quad \begin{cases} \Gamma^1_{11} = 0, & \Gamma^1_{12} = \dfrac{-2\cos v \sin v}{2\cos^2 v} = -\tan v, \\ \Gamma^1_{22} = 0, & \Gamma^2_{11} = +\dfrac{2\cos v \sin v}{2} = \sin v \cos v, \\ \Gamma^2_{12} = 0, & \Gamma^2_{22} = 0 \end{cases}$$

$\therefore$ Geodesic equations are

$$\begin{cases} u'' - 2\tan v \, u' v' = 0 & \text{—— (1)} \\ v'' + \sin v \cos v \, u'^2 = 0 & \text{—— (2)} \end{cases}$$

$$(1) \Rightarrow \quad \frac{u''}{u'} = 2\tan v \, v'$$

$$\Rightarrow \quad (\log u')' = -2(\log \cos v)'$$

$$\Rightarrow \quad \log u' = -2\log\cos v + C$$

$$\Rightarrow \quad u' = \frac{c}{\cos^2 v} \qquad \Big(\underset{\text{new}}{\underset{\uparrow}{c}} = e^{\overset{\text{old}}{C}} \Big)$$

Substitute into (2)

$$v'' + \sin v \cos v \frac{c^2}{\cos^4 v} = 0$$

$$\Rightarrow \qquad v'' + c^2 \frac{\sin v}{\cos^3 v} = 0$$

$$\Rightarrow \qquad 2v'v'' + 2c^2 \frac{\sin v}{\cos^3 v} v' = 0$$

$$\Rightarrow \qquad (v'^2)' + c^2 \left(\frac{1}{\cos^2 v} \right)' = 0$$

$$v'^2 + \frac{c^2}{\cos^2 v} = d \qquad d = \text{constant}$$

Note that $|\alpha'|^2 = \cos^2 v \, u'^2 + v'^2$

$$\therefore \qquad |\alpha'|^2 = \cos^2 v \left(\frac{c}{\cos^2 v} \right)^2 + v'^2 = d$$

Since we may reparametrize α by arc-length, we may set $d = 1$.

$$\therefore \qquad v'^2 = 1 - \frac{c^2}{\cos^2 v} \qquad \left(\Rightarrow \; c^2 \le \cos^2 v \right)$$

$$\Rightarrow \qquad v' = \pm \sqrt{\frac{\cos^2 v - c^2}{\cos^2 v}}$$

Suppose we can eliminate the parameter to write u as a function of v, then

$$\frac{du}{dv} = \frac{c}{\cos^2 v} \Big/ \pm \frac{\sqrt{\cos^2 v - c^2}}{\cos v}$$

absorbing $\pm$ into c.

$$= \pm \frac{c}{\cos v \sqrt{\cos^2 v - c^2}} = \frac{c}{\cos v \sqrt{\cos^2 v - c^2}}$$

$$\Rightarrow \qquad \int du = \int \frac{c \, dv}{\cos v \sqrt{\cos^2 v - c^2}}$$

$$= \int \frac{c\, dv}{\cos^2 v \sqrt{1 - \frac{c^2}{\cos^2 v}}}$$

$$= \int \frac{c \sec^2 v\, dv}{\sqrt{1 - c^2 \sec^2 v}}$$

$$= \int \frac{c \sec^2 v\, dv}{\sqrt{1 - c^2 - c^2 \tan^2 v}} \qquad \left(c^2 < \cos^2 v \leq 1 \right)$$

$$= \frac{c}{\sqrt{1-c^2}} \int \frac{\sec^2 v\, dv}{\sqrt{1 - \frac{c^2}{1-c^2} \tan^2 v}}$$

Let $\quad w = \frac{c}{\sqrt{1-c^2}} \tan v \quad \Rightarrow \quad dw = \frac{c}{\sqrt{1-c^2}} \sec^2 v\, dv$

$$\Rightarrow \quad u = \int \frac{dw}{\sqrt{1-w^2}}$$

Let $\quad w = \sin \theta \quad \Rightarrow \quad dw = \cos \theta\, d\theta$

$$\Rightarrow \quad u = \int d\theta = \theta + c_1$$

$$= \sin^{-1}(w) + c_1$$

$$\Rightarrow \quad \boxed{u = \sin^{-1}\left(\frac{c \tan v}{\sqrt{1-c^2}}\right) + c_1}$$

(General solution for a geodesic on S^2 in the standard coordinates.)

To see this gives a great circle, we calculate as follow:

$$u - c_1 = \sin^{-1}\left(\frac{c}{\sqrt{1-c^2}} \tan v\right)$$

$$\Rightarrow \quad \sin(u - c_1) = \frac{c}{\sqrt{1-c^2}} \tan v$$

$$\Rightarrow \quad \sin u \cos C_1 - \cos u \sin C_1 = \frac{c}{\sqrt{1-c^2}} \tan v$$

From $\mathbf{X}(u,v) = (\underbrace{\cos u \cos v}_{x}, \underbrace{\sin u \cos v}_{y}, \underbrace{\sin v}_{z})$,

we have $$\frac{y \cos C_1}{\cos v} - \frac{x \sin C_1}{\cos v} = \frac{c}{\sqrt{1-c^2}} \frac{z}{\cos v}$$

$$\Rightarrow \quad (\sin C_1)\, x + (-\cos C_1)\, y + \left(\frac{c}{\sqrt{1-c^2}}\right) z = 0$$

$\Rightarrow$ The soln. curve α lies on a plane passing through the origin (= center of S^2)

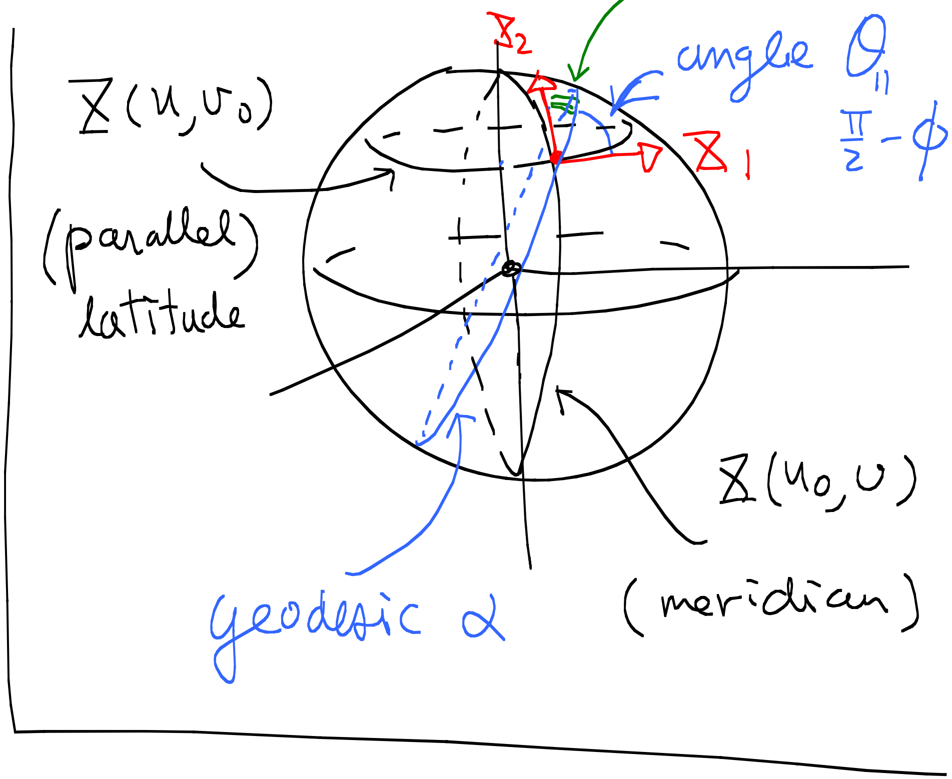
$\Rightarrow$ α is a great circle !

eg Clairaut relation for geodesic on S^2.

Let ϕ = angle between the geodesic α and the meridian $\underline{x}(u_0, v)$ at the point $\underline{x}(u_0, v_0)$.

$(0 \le \phi \le \frac{\pi}{2})$

Then $\theta = \frac{\pi}{2} - \phi$ is the angle between α and the parallel $\underline{x}(u, v_0)$ at the point $\underline{x}(u_0, v_0)$.



And we have

$$\sin\phi = \cos\theta = \frac{|\langle \alpha', \underline{x}_1 \rangle|}{|\alpha'|\,|\underline{x}_1|} = \frac{|\langle u'\underline{x}_1 + v'\underline{x}_2, \underline{x}_1 \rangle|}{|\underline{x}_1|}$$

$$= |u'|\,|\underline{x}_1|$$

$$\Rightarrow \quad \sin\phi = \cos\theta = \frac{c}{\cos^2 v_0} \cos v_0$$

$$\Rightarrow \quad \boxed{\sin\phi = \cos\theta = \frac{c}{\cos v_0}}$$

Note that $\cos v_0$ = <u>radius</u> r of the <u>parallel</u> (latitude)

we have

$$\boxed{r \sin\phi = c}$$

(This is the Clairaut's relation in Oprea)

(latitude) radius of the parallel at the intersection pt.

angle between the geodesic & the meridian at the intersection pt.

constant.

or

$$\boxed{r\cos\theta = c}$$

radius of the (latitude) parallel at the intersection pt.

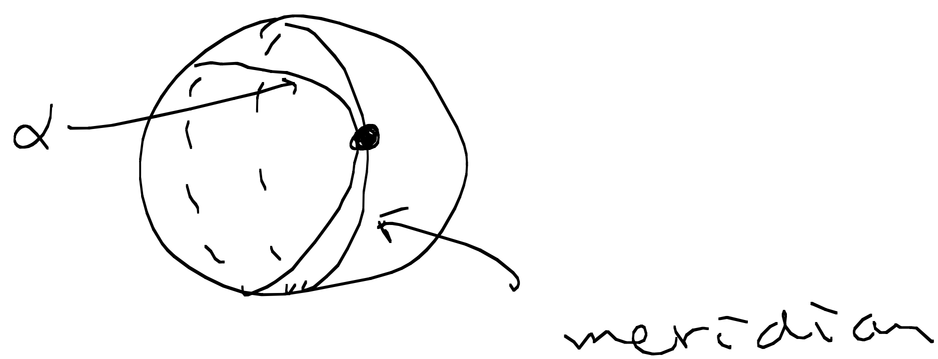
angle between the geodesic & the parallel (latitude) at the intersection pt.

constant.

(This is the Clairaut's relation as in do Carmo.)

Application: (Reading Exercise)

(1) If a geodesic α tangent to a meridian at some point.



Then $\phi = 0$ at this point & hence

$$c = r \sin 0 = 0 .$$

Therefore, Clairaut's relation implies that

$$r \sin \phi = 0 \quad \text{at all point}$$

$$\Rightarrow \quad \phi \equiv 0$$

$\Rightarrow$ α is a meridian

(This is consistence with the uniqueness of geodesic with initial point & tangent vector.)

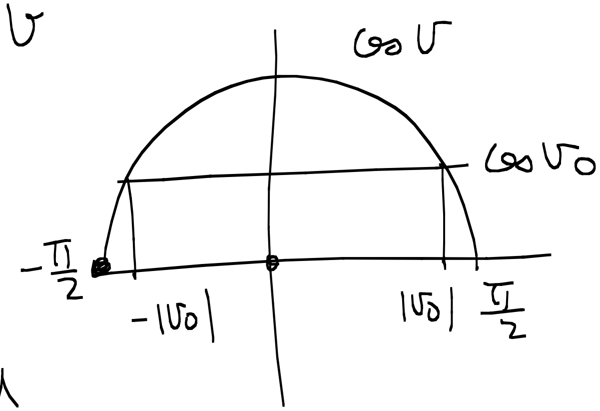
(2) If a geodesic α tangent to a parallel (latitude) $\gamma(u, v_0)$ at some point. Then at that point

$\phi = \frac{\pi}{2}$, $r = \cos v_0 \Rightarrow c = \cos v_0$

Clairaut's relation $\Rightarrow$

$$r \sin\phi = \cos v_0 \quad \text{along } \alpha .$$

$$\Rightarrow \quad \cos v_0 \le r = \cos v$$



$$\Rightarrow \quad -|v_0| \le v \le |v_0|$$

$\Rightarrow$ The geodesic is bounded between the parallels (latitudes)

$\mathbf{X}(u, |v_0|)$ & $\mathbf{X}(u, -|v_0|)$

eg <u>Torus</u> T^2 with parametrization

$$X(u,v) = \big((R+r\cos u)\cos v,\ (R+r\cos u)\sin v,\ r\sin u\big)$$

$$\Rightarrow \quad X_1 = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u)$$

$$X_2 = (-(R+r\cos u)\sin v,\ (R+r\cos u)\cos v,\ 0)$$

$$\Rightarrow \quad (g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & (R+r\cos u)^2 \end{pmatrix} \qquad (\text{indep of } v)$$

$$\left\{ \begin{aligned} &\Gamma^1_{11} = 0, \quad \Gamma^1_{12} = 0, \\ &\Gamma^1_{22} = -\frac{2(R+r\cos u)(-r\sin u)}{2r^2} = \frac{(R+r\cos u)\sin u}{r} \end{aligned} \right.$$

$$\begin{cases} \Gamma^2_{11} = 0, \quad \Gamma^2_{12} = -\dfrac{(R+r\cos u)\, r\sin u}{(R+r\cos u)^2} = \dfrac{-r\sin u}{R+r\cos u} \\[2ex] \Gamma^2_{22} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} u'' + \dfrac{(R+r\cos u)\sin u}{r}\, v'^2 = 0 \quad \text{—— (1)} \\[2ex] v'' - \dfrac{2r\sin u}{R+r\cos u}\, u'v' = 0 \quad \text{—— (2)} \end{cases}$$

$$(2) \Rightarrow \quad \frac{v''}{v'} = \frac{2r\sin u\, u'}{R+r\cos u}$$

$$\Rightarrow \quad (\log v')' = -2\left(\log(R+r\cos u)\right)'$$

$$\Rightarrow \quad v' = \frac{c}{(R+r\cos u)^2} \qquad (c = \text{const.})$$

We may substitute this into (1) to use similar method as in eg for S^2. Or we can assume now α is of <u>unit speed</u> and conclude

$$1 = r^2 u'^2 + (R + r\cos u)^2 v'^2$$

$$= r^2 u'^2 + \frac{c^2}{(R+r\cos u)^2} \qquad \left(\Rightarrow c^2 \le (R+r\cos u)^2\right)$$

$$\Rightarrow \quad u' = \pm\frac{1}{r}\sqrt{1 - \frac{c^2}{(R+r\cos u)^2}}$$

Then

$$\frac{dv}{du} = \frac{\dfrac{c}{(R+r\cos u)^2}}{\pm\frac{1}{r}\sqrt{1-\dfrac{c^2}{(R+r\cos u)^2}}}$$

$$\Rightarrow \quad \frac{dv}{du} = \frac{cr}{(R+r\cos u)\sqrt{(R+r\cos u)^2 - c^2}}$$

(by absorbing $\pm$ into c)

$$\Rightarrow \quad \boxed{v = \int \frac{cr\,du}{(R+r\cos u)\sqrt{(R+r\cos u)^2 - c^2}}}$$

which cannot be integrated explicitly.

Clairaut's relation for Torus

The parallels are

$$X(u_0, v) = ((R+r\cos u_0)\cos v, (R+r\cos u_0)\sin v, r\sin u_0)$$

$\Rightarrow$ ϕ = angle between α & a meridian at the intersection point

is given by

$$\sin\phi = \frac{|\langle u' \mathbf{x}_1 + v' \mathbf{x}_2, \mathbf{x}_2 \rangle|}{|\alpha'|\, |\mathbf{x}_2|}$$

$$= |v'|\, |\mathbf{x}_2|$$

$$\therefore \quad \sin\phi = \frac{c}{(R + r\cos u_0)^2} \cdot (R + r\cos u_0)$$

$$\Rightarrow \quad \boxed{(R + r\cos u_0)\sin\phi = c}$$

(again this is the radius of the parallel $\mathbf{x}(u_0, v)$)

Def: A patch $\mathbf{x}(u,v)$ is called a

- Clairaut parametrization in u if

$$(g_{ij}) = \begin{pmatrix} g_{11}(u) & 0 \\ 0 & g_{22}(u) \end{pmatrix},$$ i.e. orthogonal & independent of v

- Clairaut parametrization in v if

$$(g_{ij}) = \begin{pmatrix} g_{11}(v) & 0 \\ 0 & g_{22}(v) \end{pmatrix},$$ i.e. orthogonal & independent of u

In classical notation,

$$\begin{pmatrix} E(u) & 0 \\ 0 & G(u) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} E(v) & 0 \\ 0 & G(v) \end{pmatrix}$$

In these cases, we have

u- Clairaut geodesic equation

$$\begin{cases} u'' + \dfrac{E_u}{2E} u'^2 - \dfrac{G_u}{2E} v'^2 = 0 \\ v'' + \dfrac{G_u}{G} u'v' = 0 \end{cases}$$

or v- Clairaut geodesic equation

$$\begin{cases} u'' + \dfrac{E_v}{E} u'v' = 0 \\ v'' - \dfrac{E_v}{2G} u'^2 + \dfrac{G_v}{2G} v'^2 = 0 \end{cases}$$

Let consider the u-Clairaut geodesic equations:

$$\begin{cases} u'' + \dfrac{E_u}{2E} u'^2 - \dfrac{G_u}{2E} v'^2 = 0 & \text{—— (1)} \\ v'' + \dfrac{G_u}{G} u'v' = 0 & \text{—— (2)} \end{cases}$$

$$(2) \Rightarrow \quad \frac{v''}{v'} = -\frac{G_u}{G} u' = -(\log G)' \qquad (G = G(u))$$

$$\Rightarrow \quad (\log v')' = -(\log G)'$$

$$\Rightarrow \quad v' = \frac{c}{G(u)} \quad \text{for some constant } c$$

Unit speed condition $\Rightarrow$ $1 = E(u) u'^2 + G(u) v'^2$

$$= E(u) u'^2 + \frac{c^2}{G(u)}$$

$$\Rightarrow \quad u' = \pm \frac{1}{\sqrt{E(u)}} \sqrt{1 - \frac{c^2}{G(u)}}$$

$$\Rightarrow \quad \frac{dv}{du} = \frac{\frac{c}{G(u)}}{\pm \frac{1}{\sqrt{E(u)}} \sqrt{1 - \frac{c^2}{G(u)}}} = \frac{c\sqrt{E(u)}}{\sqrt{G(u)(G(u) - c^2)}}$$

($\pm$ absorbed into c)

$$\Rightarrow \boxed{v = \int \frac{c\sqrt{E(u)}\,du}{\sqrt{G(u)\,(G(u)-c^2)}}}$$

is a solution of geodesic.

Similarly, we have a solution for v-Clairaut geodesic equations

$$\boxed{u = \int \frac{c\sqrt{G(v)}\,dv}{\sqrt{E(v)\,(E(v)-c^2)}}}$$

Prop (Prop 5.2.7 of Oprea) On a surface with u-Clairaut parametrization, u-parameter curves are geodesics. (when reparametrized to have constant speed.)

Pf: A u-parameter curve $\alpha(t)$ is given by $\alpha(t) = X(t, v_0)$, $v_0 =$ constant.

$$\Rightarrow \text{arc-length } s(t) = \int_0^t |\alpha'(t)|\,dt = \int_0^t |X_1(t)|\,dt$$

$$= \int_0^t \sqrt{E(t)}\,dt$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{E(t)} > 0.$$

$\Rightarrow$ Inverse $t = t(s)$ exists.

Let $\beta(s) = \alpha(t(s)) = X(t(s), v_0)$

i.e. $\begin{cases} u = t(s) \\ v = v_0 \end{cases}$

$$\Rightarrow \quad \frac{du}{ds} = \frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{\sqrt{E(t(s))}}$$

$$\frac{d^2u}{ds^2} = -\frac{1}{2\,E(\alpha(s))^{3/2}} \cdot E_u(\alpha(s)) \cdot \frac{d\alpha}{ds}$$

$$= -\frac{1}{2E^{3/2}} E_u \cdot \frac{1}{\sqrt{E}}$$

$$= -\frac{E_u}{2E^2}.$$

Together with $v' = v'' = 0$, we have

$$u'' + \frac{E_u}{2E} u'^2 - \frac{G_u}{2E} v'^2 = \frac{d^2u}{ds^2} + \frac{E_u}{2E}\left(\frac{du}{ds}\right)^2$$

$$= -\frac{E_u}{2E^2} + \frac{E_u}{2E}\left(\frac{1}{\sqrt{E}}\right)^2 = 0$$

The second eqt. is trivial & hence we've proved that u-parameter curves (parametrized by arc-length) are geodesics. ※

Remark: One can also study v-parameter curves in u-Clairaut parametrization: $u = u_0$, $v = t$.

Geodesic eqts is satisfied $\iff$ $-\dfrac{G_u}{2E} = 0$ i.e. $\boxed{G_u(u_0) = 0}$

Combining with previous result:

Thm (Thm 5.2.8 of Oprea)

On a surface with u-Clairaut coordinates $\mathbf{x}(u,v)$

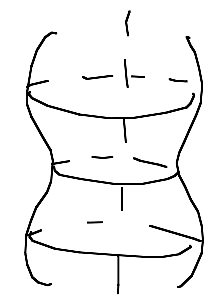
- u-parameter curves are geodesics (after reparametrization)
- a v-parameter curve $\mathbf{x}(u_0, v)$ is a geodesic $\iff$ $G_u(u_0) = 0$.

Application :

Surface of revolution given by

$$\mathbf{X}(u,v) = \big(g(u),\ h(u)\cos v,\ h(u)\sin v\big).$$

Then
$$\mathbf{X}_1 = (g',\ h'\cos v,\ h'\sin v)$$
$$\mathbf{X}_2 = (0,\ -h\sin v,\ h\cos v)$$



$$\Rightarrow \quad (g_{ij}) = \begin{pmatrix} g'(u)^2 + h'(u)^2 & 0 \\ 0 & h(u)^2 \end{pmatrix}$$

is u- Clairaut

$\Rightarrow$ $\begin{cases} u\text{- parameter curves } (= \text{meridians}) \text{ are geodesics} \\ v\text{- parameter curves } (= \text{parallels}) \text{ are geodesics} \end{cases}$

$\Leftrightarrow$ $h' = 0$ at the pt.

(This is cor. 5.2.9 of the ref. book !)

Putting $E(u) = g'(u)^2 + h'(u)^2$

& $G(u) = h^2(u)$

We see that geodesics on the surface of revolution satisfies

$$v = \int \frac{c\,(g'(u)^2 + h'(u)^2)\,du}{\sqrt{h^2(u)\,(h^2(u) - c^2)}}$$

If the meridians are parametrized by arc-length, then $g'(u)^2 + h'(u)^2 = 1$ and

we have $$\boxed{v = \int \frac{c\,du}{h(u)\sqrt{h(u)^2 - c^2}}}$$

(Application to a physical system of Clairaut relation is omitted.)

5.3 A Brief Digression on Completeness.

<u>Def</u> : A surface M is <u>geodesically complete</u> if every (unit speed) geodesic has domain $\mathbb{R} = (-\infty, \infty)$

(Note: ODE theory only implies existence of geodesic defined on an open interval $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small.)

<u>egs</u> $\mathbb{R}^2$ is a (geodesically) complete surface in $\mathbb{R}^3$ but

$\mathbb{R}^2 \setminus \{(0,0)\}$ is <u>not</u>.

($\underbrace{\text{the geodesic}}_{\text{(unit speed)}}$ $s \in (0, +\infty) \longmapsto (s, 0, 0)$ cannot be extended to a geodesic defined on $(-\infty, +\infty) = \mathbb{R}$.)

Hopf-Rinow Thm (Thm 5.3.1 of Oprea)

If M is geodesically complete, then any 2 points of M may be joined by a geodesic which has the shortest length of any curve between the 2 points

(Pf: omitted)

Thm (Thm 5.3.2 of Oprea)

A closed surface $M \subseteq \mathbb{R}^3$ is geodesically complete

(Pf: Omitted)

(in the topological sense: closed subset of $\mathbb{R}^3$)

§5.4 Surfaces not in $\mathbb{R}^3$

(Omitted)

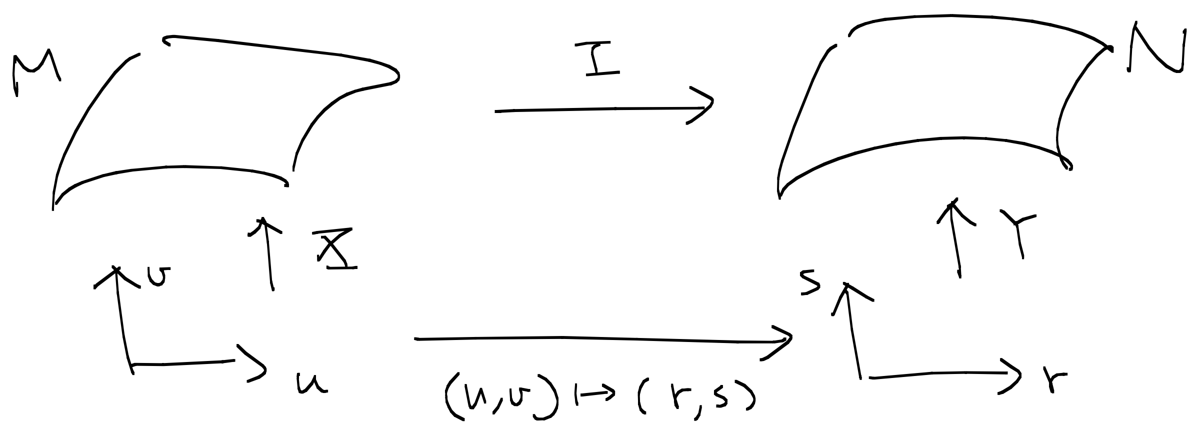
5.5 Isometries and Conformal Maps

Let $M: \mathbf{X}(u,v)$ and $N: Y(r,s)$ be 2 surfaces given by the patch $\mathbf{X}(u,v)$ & $Y(r,s)$ respectively.

Suppose $I: M \to N$ is a map defined by

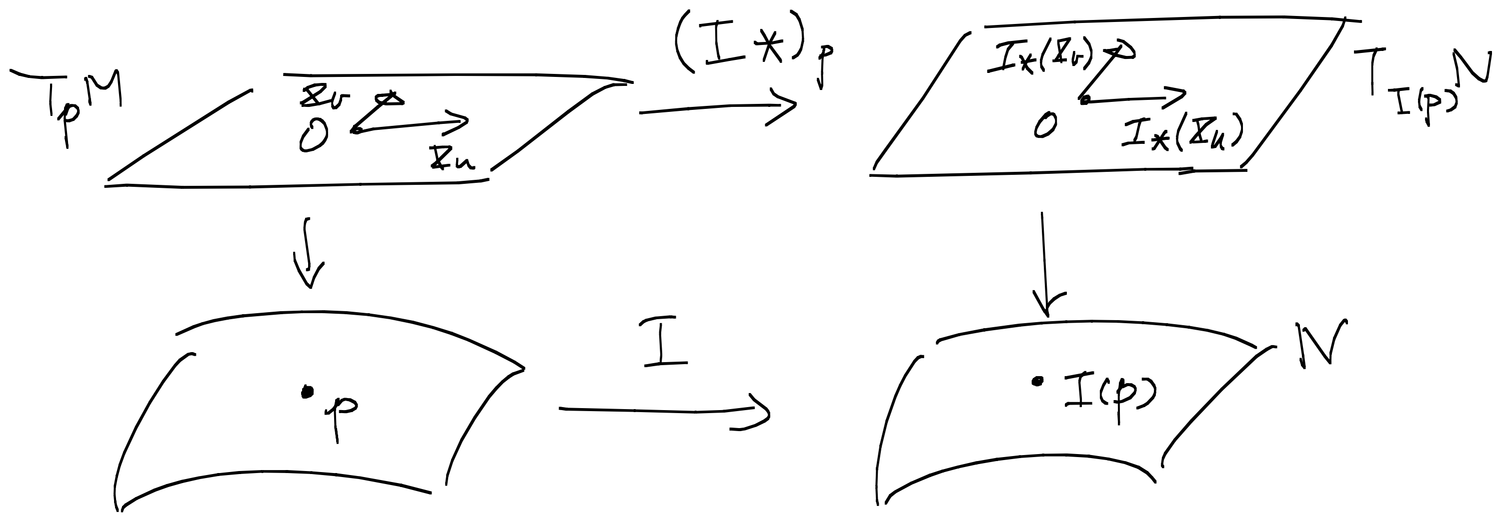
$$(u,v) \longmapsto (r(u,v),\, s(u,v))$$

Or more precisely $I(\mathbf{X}(u,v)) = Y(r(u,v), s(u,v))$



Let $I_* = DI$ be the differential of I.

Then $(I_*)_p : T_pM \to T_{I(p)}N$



<u>Def</u>: A map $I : M \to N$ is called a <u>(local) isometry</u>

if $\forall p$
$$\begin{cases} \langle I_*(X_u), I_*(X_u) \rangle_{I(p)} = \langle X_u, X_u \rangle_p \\ \langle I_*(X_u), I_*(X_v) \rangle_{I(p)} = \langle X_u, X_v \rangle_p \\ \langle I_*(X_v), I_*(X_v) \rangle_{I(p)} = \langle X_v, X_v \rangle_p . \end{cases}$$

Equivalent to $\langle I_*(V), I_*(W) \rangle_{I(p)} = \langle V, W \rangle_p$, $\forall V, W \in T_pM$. $\forall p$

(Oprea only defines isometry by assuming $\langle X_u, X_v \rangle = \langle I_*(X_u), I_*(X_v) \rangle = 0$ which is unnatural.)

Remark : Let γ be a curve on M, then $I(\gamma)$ is a curve on N.

Then
$$L(I(\gamma)) = \int_a^b \sqrt{\left\langle \frac{d}{dt}(I(\gamma)), \frac{d}{dt}(I(\gamma)) \right\rangle_N}\, dt$$

$$= \int_a^b \sqrt{\langle I_*(\gamma'), I_*(\gamma') \rangle_N}\, dt$$

$I =$ isometry $\longrightarrow$
$$= \int_a^b \sqrt{\langle \gamma', \gamma' \rangle_M}\, dt = L(\gamma)$$

$\therefore$ Isometry preserves length.

Remark : Isometry also preserves angles as angle θ between 2 tangent vectors V, W is given by

$$\cos\theta = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle}\sqrt{\langle W, W \rangle}}.$$

In <u>modern notation</u>, we write

$$ds^2 = \sum g_{ij}\, du^i \otimes du^j \quad , \text{ where } \quad g_{ij} = \langle X_i, X_j \rangle$$

for the metric, or the 1^{st} fundamental form, on M in the coordinate $X(u,v)$, $(u^1 = u,\ u^2 = v)$.

If we let $(\bar{u}^1, \bar{u}^2) = (r, s)$ for coordinate $Y(r,s)$ on N

$$\bar{g}_{kl} = \langle Y_k, Y_l \rangle$$ be the metric coefficients of N.

Then the metric on N can be written as

$$d\bar{s}^2 = \sum_{k,l} \bar{g}_{kl}(\bar{u}^1, \bar{u}^2)\, d\bar{u}^k \otimes d\bar{u}^l$$

Then for map $I : M \to N$, we define

$$I^*(d\bar{s}^2) = \sum_{k,l} \bar{g}_{kl}(I(u^1,u^2)) \left(\sum_i \frac{\partial \bar{u}^k}{\partial u^i} du^i \right) \otimes \left(\sum_j \frac{\partial \bar{u}^l}{\partial u^j} du^j \right)$$

(pull-back of $d\bar{s}^2$ by I)

$$= \sum_{i,j} \left(\sum_{k,l} \bar{g}_{kl}(I(u^1,u^2)) \frac{\partial \bar{u}^k}{\partial u^i} \frac{\partial \bar{u}^l}{\partial u^j} \right) du^i \otimes du^j$$

(We've used $(\bar{u}^1, \bar{u}^2) = I(u^1, u^2)$ to denote the map $I : M \to N$ for simplicity.)

Claim: the coefficients of $I^*(d\bar{s}^2)$ and $\langle I_*(X_i), I_*(X_j) \rangle$ coincide and hence

$$\boxed{\begin{aligned} &I \text{ is an isometry} \\ \Longleftrightarrow \quad &\sum_{k,l} \bar{g}_{kl} \frac{\partial \bar{u}^k}{\partial u^i} \frac{\partial \bar{u}^l}{\partial u^j} = g_{ij} \\ \Longleftrightarrow \quad &I^*(d\bar{s}^2) = ds^2 \end{aligned}}$$

Pf: By definition $I_*(X_1) = \left.\frac{d}{dt}\right|_{t=0} I(X(u^1+t, u^2))$

$$= \left.\frac{d}{dt}\right|_{t=0} Y(\bar{u}^1(u^1+t, u^2), \bar{u}^2(u^1+t, u^2))$$

$$= Y_1 \frac{\partial \bar{u}^1}{\partial u^1} + Y_2 \frac{\partial \bar{u}^2}{\partial u^1} \quad \left(\begin{array}{l} Y_1, Y_2 \text{ evaluate at} \\ (\bar{u}^1(u^1,u^2), \bar{u}^2(u^1,u^2)) \end{array} \right)$$

$$= \sum_k \frac{\partial \bar{u}^k}{\partial u^1} Y_k(I(u^1,u^2))$$

Similarly, we have $I_*(X_i) = \sum_k \frac{\partial \bar{u}^k}{\partial u^i} Y_k$

Hence $\langle I_*(X_i), I_*(X_j) \rangle = \left\langle \sum_k \frac{\partial \bar{u}^k}{\partial u^i} Y_k, \sum_l \frac{\partial \bar{u}^l}{\partial u^j} Y_l \right\rangle$

$$= \sum_{k,l} \langle Y_k, Y_l \rangle \frac{\partial \bar{u}^k}{\partial u^i} \frac{\partial \bar{u}^l}{\partial u^j}$$

$$= \sum_{k,l} \bar{g}_{kl} \frac{\partial \bar{u}^k}{\partial u^i} \frac{\partial \bar{u}^l}{\partial u^j} \quad \left(\bar{g}_{kl} = \bar{g}_{kl}(I(u^1,u^2)) \right)$$

#

In the notation of Oprea and under the condition

$F = g_{12} = 0$ & $\overline{F} = \overline{g}_{12} = 0$, we must have

$$E = \overline{E}\left(\frac{\partial r}{\partial u}\right)^2 + \overline{G}\left(\frac{\partial s}{\partial u}\right)^2$$

$$0 = \overline{E}\,\frac{\partial r}{\partial u}\frac{\partial r}{\partial v} + \overline{G}\,\frac{\partial s}{\partial u}\frac{\partial s}{\partial v}$$ $\leftarrow$ this is assumed by Oprea

$$G = \overline{E}\left(\frac{\partial r}{\partial v}\right)^2 + \overline{G}\left(\frac{\partial s}{\partial v}\right)^2$$

these are the equations given in Oprea.

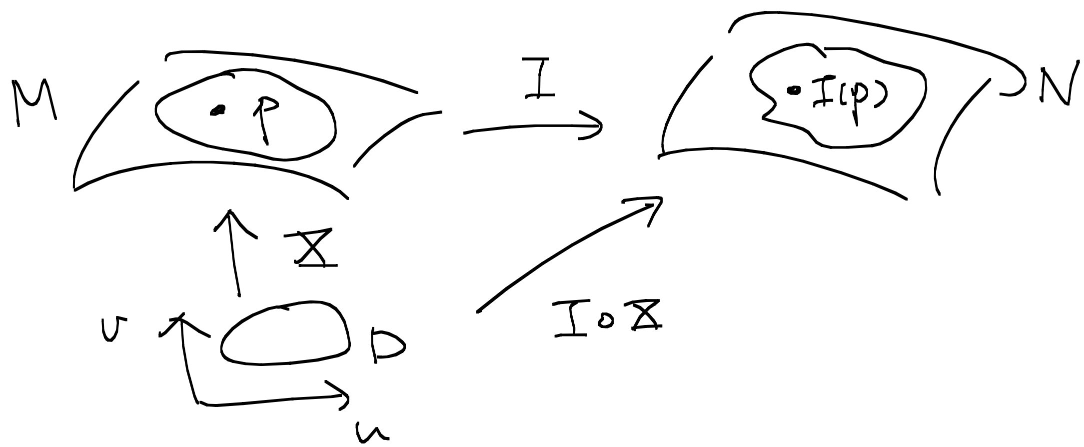
Thm (Thm 5.5.1 of Oprea)

If $I : M \to N$ is an isometry, then the Gauss curvatures at corresponding points are equal. i.e. $K_M(p) = K_N(I(p))$, $\forall p \in M$.

Pf : Since I is an isometry, $(I_*)_p : T_pM \to T_{I(p)}M$, $\forall p$, is a nonsingular linear map. Hence Inverse Function Theorem $\Rightarrow$ I is locally invertible. Therefore

$$I \circ X : D \to N$$

is a coordinate patch for N around the point $I(p)$, where $X : D \to M$ is a patch for M around p.



Then I is an isometry $\Rightarrow$ the metric coefficients of N calculated in the coordinate patch $I \circ X$ are the same as M. Therefore, Gauss' Theorma Egregium $\Rightarrow$ $K_M(p) = K_N(I(p))$. ⌘

Remark: If M, N are parametrized by the same domain

$(u,v) \in D$: $X : D \to M$ & $Y : D \to N$.

Suppose $I : M \to N$ is the map defined by

$$I(X(u,v)) = Y(u,v).$$

Namely, at the parameters level, $I : (u,v) \mapsto (u,v)$

i.e. $\bar{u}^i = u^i \quad \forall i$

$$\Rightarrow \quad \frac{\partial \bar{u}^k}{\partial u^i} = \delta_i^k$$

and hence I is an isometry

$$\Leftrightarrow \quad g_{ij} = \sum_{k,l} \bar{g}_{kl} \frac{\partial \bar{u}^k}{\partial u^i} \frac{\partial \bar{u}^l}{\partial u^j}$$

$$= \sum_{k,l} \bar{g}_{kl} \delta_i^k \delta_j^l = \bar{g}_{ij}$$

$\therefore$ $I : M \to N$ is an isometry $\Leftrightarrow$ M & N have the same metric coefficients.

Def: Let $I: M \to N$ be a local isometry. We say that I is a <u>global isometry</u> if $I^{-1}: N \to M$ exists.

eg: (Helicoid)

Let $M: X(u,v) = (u\cos v, u\sin v, v)$ and

$N: Y(r,s) = (\sinh r \cos s, \sinh r \sin s, s)$.

Then $\begin{cases} X_1 = (\cos v, \sin v, 0) \\ X_2 = (-u\sin v, u\cos v, 1) \end{cases}$

$$\Rightarrow \quad g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1 + u^2$$

And $\begin{cases} Y_1 = (\cosh r \cos s, \cosh r \sin s, 0) \\ Y_2 = (-\sinh r \sin s, \sinh r \cos s, 1) \end{cases}$

$$\Rightarrow \quad \overline{g}_{11} = \cosh^2 r, \quad \overline{g}_{12} = 0, \quad \overline{g}_{22} = 1 + \sinh^2 r.$$

Define $I : M \to N$ by

$$\begin{cases} r = \sinh^{-1} u \\ s = v \end{cases} \quad \text{i.e.} \quad \begin{cases} u = \sinh r \\ v = s \end{cases}$$

Then
$$\begin{cases} \dfrac{\partial r}{\partial u} = \dfrac{1}{\sqrt{1+u^2}}, & \dfrac{\partial r}{\partial v} = 0 \\[2ex] \dfrac{\partial s}{\partial u} = 0 \quad , & \dfrac{\partial s}{\partial v} = 1 \end{cases}$$

$$\therefore \quad I^*(d\bar{s}^2)_{11} = \bar{g}_{11}\left(\frac{\partial r}{\partial u}\right)^2 + 2\bar{g}_{12}\frac{\partial r}{\partial u}\frac{\partial s}{\partial u} + \bar{g}_{22}\left(\frac{\partial s}{\partial u}\right)^2$$

$$= \cosh^2 r \cdot \frac{1}{1+u^2} + 0 + (1+\sinh^2 r)\cdot 0$$

$$= 1 = g_{11} \qquad (\text{since } 1+u^2 = 1+\sinh^2 r = \cosh^2 u)$$

$$I^*(d\bar{s}^2)_{12} = \bar{g}_{11}\left(\frac{\partial r}{\partial u}\right)\left(\cancel{\frac{\partial r}{\partial v}}\right) + \cancel{\bar{g}_{12}}\frac{\partial r}{\partial u}\frac{\partial s}{\partial v}$$

$$+ \cancel{\bar{g}_{21}}\frac{\partial r}{\partial v}\frac{\partial s}{\partial u} + \bar{g}_{22}\cancel{\frac{\partial s}{\partial u}}\frac{\partial s}{\partial v}$$

$$= 0 = g_{12}$$

$$I^*(d\bar{s}^2)_{22} = \bar{g}_{11}\left(\cancel{\frac{\partial r}{\partial v}}\right)^2 + 2\,\bar{g}_{12}\,\cancel{\frac{\partial r}{\partial v}}\,\frac{\partial s}{\partial v} + \bar{g}_{22}\left(\frac{\partial s}{\partial v}\right)^2$$

$$= (1+\sinh^2 r)\cdot 1$$

$$= 1+u^2 = g_{22}$$

$$\therefore \quad I^*(d\bar{s}^2) = ds^2$$

$\Rightarrow$ I is an isometry

Clearly, I^{-1} exists, $\therefore$ I is in fact a global isometry.

<u>Thm</u> (Thm 5.5.8 of Oprea)

Let $\alpha(t)$ be a geodesic on a surface M and let $I: M \to N$ be an isometry. Then the curve $\beta(t) = I(\alpha(t))$ is an geodesic of N.

Pf: Similar to the proof of $K_M(p) = K_N(I(p))$. $\#$

Conformal Maps.

Def: A map $I : M \to N$ is called conformal if $\boxed{I^* d\bar{s}^2 = \lambda^2 ds^2}$ for some nonvanishing function λ on M

eg: Isometries are conformal, $\lambda \equiv 1$.

Remarks: • $I^* d\bar{s}^2 = \lambda^2 ds^2$

$$\Longleftrightarrow \quad \langle I_*(X_i), I_*(X_j) \rangle_{I(p)} = \lambda^2(p) \langle X_i, X_j \rangle_p$$

$$\Longleftrightarrow \quad \langle I_*(V), I_*(W) \rangle_{I(p)} = \lambda^2(p) \langle V, W \rangle_p, \quad \forall\, V, W \in T_pM.$$

- Conformal map preserves angles :

$$\frac{\langle I_*(V), I_*(W)\rangle_{I(p)}}{\sqrt{\langle I_*(V), I_*(V)\rangle_{I(p)}}\sqrt{\langle I_*(W), I_*(W)\rangle_{I(p)}}}$$

$$= \frac{\lambda^2 \langle V, W\rangle_p}{\sqrt{\lambda^2\langle V, V\rangle_p}\sqrt{\lambda^2\langle W, W\rangle_p}} = \frac{\langle V, W\rangle_p}{\sqrt{\langle V, V\rangle_p}\sqrt{\langle W, W\rangle_p}} . \#$$

Ch 6 Holonomy and the Gauss-Bonnet Theorem

(6.1 Introduction : See Oprea)

6.2 Covariant Derivative

Let $\mathbf{X}(u,v)$ be a coordinate patch on a surface $M \subset \mathbb{R}^3$

such that

$$g_{12} = F = 0.$$

Then

$$E_1 = \frac{1}{\sqrt{g_{11}}}\mathbf{X}_1, \quad E_2 = \frac{1}{\sqrt{g_{22}}}\mathbf{X}_2$$

satisfy

$$\begin{cases} \langle E_1, E_1 \rangle = \langle \frac{1}{\sqrt{g_{11}}}\mathbf{X}_1, \frac{1}{\sqrt{g_{11}}}\mathbf{X}_1 \rangle = \frac{1}{g_{11}}\langle \mathbf{X}_1, \mathbf{X}_1 \rangle = 1 \\ \langle E_1, E_2 \rangle = 0 \\ \langle E_2, E_2 \rangle = 1 \end{cases}$$

i.e. $\boxed{\langle \xi_i, \xi_j \rangle = \delta_{ij}}$

$\{\xi_1, \xi_2\}$ is called a <u>moving frame</u> on the patch $X(u,v)$.

Now, let $\alpha(t)$ be a (closed) curve with <u>constant speed</u>

i.e. $|\alpha'| = \text{const}$.

Consider <u>vector fields along α</u> : $\begin{cases} \xi_1(t) = \xi_1(\alpha(t)) \\ \xi_2(t) = \xi_2(\alpha(t)) \end{cases}$,

(see do Carmo for the definition of general vector fields along curves.)

and the directional derivative

$$\nabla^{\mathbb{R}^3}_{\alpha'} \xi_i = \frac{d}{dt} \xi_i(\alpha(t)).$$

Since $\{E_1, E_2\}$ is an orthonormal basis for T_pM, $\forall p \in M$, we have

$$\boxed{\nabla^{\mathbb{R}^3}_{\alpha'} E_i = \sum_j \omega_{ji} E_j + s_i U}$$

for some functions $\omega_{ji}(t)$ & $s_i(t)$, where U = surface normal.

Then $0 = \alpha'(\langle E_i, E_i \rangle)$

$$= 2\langle \nabla^{\mathbb{R}^3}_{\alpha'} E_i, E_i \rangle$$

$$= 2 \langle \sum_j \omega_{ji} E_j + s_i U, E_i \rangle$$

$$= 2\omega_{ii}$$

$\therefore$ $\omega_{11} = \omega_{22} = 0$ and $\begin{cases} \nabla^{\mathbb{R}^3}_{\alpha'} E_1 = \omega_{21} E_2 + s_1 U \\ \nabla^{\mathbb{R}^3}_{\alpha'} E_2 = \omega_{12} E_1 + s_2 U \end{cases}$

Furthermore,
$$\begin{aligned} 0 &= \alpha'(\langle \xi_1, \xi_2 \rangle) \\ &= \langle \nabla^{\mathbb{R}^3}_{\alpha'} \xi_1, \xi_2 \rangle + \langle \xi_1, \nabla^{\mathbb{R}^3}_{\alpha'} \xi_2 \rangle \\ &= \langle \omega_{21} \xi_2 + s_1 U, \xi_2 \rangle + \langle \xi_1, \omega_{12} \xi_1 + s_2 U \rangle \\ &= \omega_{21} + \omega_{12} \end{aligned}$$

Therefore
$$\begin{cases} \nabla^{\mathbb{R}^3}_{\alpha'} \xi_1 = \omega_{21} \xi_2 + s_1 U \\ \nabla^{\mathbb{R}^3}_{\alpha'} \xi_2 = -\omega_{21} \xi_1 + s_2 U \end{cases}$$

Let denote
$$\nabla_{\alpha'} \xi_i = (\nabla^{\mathbb{R}^3}_{\alpha'} \xi_i)_{tan}$$
the tangential part of $\nabla^{\mathbb{R}^3}_{\alpha'} \xi_i$. Then we have
$$\begin{cases} \nabla_{\alpha'} \xi_1 = \omega_{21} \xi_2 \\ \nabla_{\alpha'} \xi_2 = -\omega_{21} \xi_1 \end{cases}$$

Note: ω_{21} depends on α', should be denoted by $\omega_{21}(\alpha')$ when needed.

Note that $\nabla_{\alpha'} \mathcal{E}_i$ is a tangent vector field along $\alpha(t)$ and is called the <u>covariant derivative</u> of the vector field $\mathcal{E}_i$ along α.

<u>Local expression of $\nabla_{\alpha'} \mathcal{E}_i$</u> :

For simplicity, write $V = \mathcal{E}_i$ and suppose

$$V(t) = a^1(t) X_1 + a^2(t) X_2 = \sum_i a^i(t) X_i$$

Then

$$\nabla_{\alpha'} V = \left[\nabla^{\mathbb{R}^3}_{\alpha'} \left(\sum_i a^i X_i \right) \right]_{\tan}$$

$$= \sum_i \left[\left(\nabla^{\mathbb{R}^3}_{\alpha'} a^i \right) X_i + a^i \nabla^{\mathbb{R}^3}_{\alpha'} X_i \right]_{\tan}$$

Note that $\nabla^{\mathbb{R}^3}_{\alpha'} a^i = \dfrac{da^i}{dt}$ (is just the directional derivative of a^k)

Let $\alpha(t) = X(u^1(t), u^2(t))$.

Then
$$\nabla^{R^3}_{\alpha'} X_i = \frac{d}{dt} X_i(u^1(t), u^2(t))$$
$$= \sum_j X_{ij} \frac{du^j}{dt}$$
$$= \sum_j \Big(\sum_k \Gamma^k_{ij} X_k + \overbrace{h_{ij} U}^{\text{normal}} \Big) \frac{du^j}{dt}$$

$$\therefore \quad \nabla_{\alpha'} V = \sum_i \frac{da^i}{dt} X_i + \sum_i a^i \sum_{j,k} \Gamma^k_{ij} X_k \frac{du^j}{dt}$$

$$\Rightarrow \quad \boxed{\nabla_{\alpha'} V = \sum_k \left[\frac{da^k}{dt} + \sum_{i,j} a^i \Gamma^k_{ij} \frac{du^j}{dt} \right] X_k}$$

This formula in fact defines <u>covariant derivative</u> of any vector field along α:

Def: Let $V = \sum_i a^i(t) X_i$ be a (differentiable) vector field along $\alpha(t) = X(u^1(t), u^2(t)) \subset M$. The expression

$$\nabla_{\alpha'} V = \sum_k \left[\frac{da^k}{dt} + \sum_{i,j} \Gamma^k_{ij} \frac{du^j}{dt} a^i \right] X_k$$

is well-defined and is called the <u>covariant derivative</u> of $V(t)$ at t.

(In the reference book by do Carmo, $\nabla_{\alpha'} W$ is denoted by $\frac{DV}{dt}$.)

Furthermore, if V is a vector field on M and $v \in T_pM$ is a tangent vector at p. Then the <u>covariant derivative</u> of V wrt v is defined by

$$\nabla_v V = \nabla_{\alpha'} V(0)$$

where α is any curve s.t. $\alpha(0) = p$ & $\alpha'(0) = v$.

Check this is well defined

Prop (Ex. 6.2.3 of Oprea)

(1) $\nabla_{f\alpha' + g\beta'} Z = f \nabla_{\alpha'} Z + g \nabla_{\beta'} Z$

(2) $\nabla_{\alpha'} (f Z) = \frac{d}{dt} f(\alpha(t)) Z + f \nabla_{\alpha'} Z$

(3) If $X(u,v)$ is an orthogonal patch, ie $g_{12} = 0$, then

$$\nabla_{X_1} X_2 = \nabla_{X_2} X_1 .$$

(4) $\frac{d}{dt} \langle W, V \rangle = \langle \nabla_{\alpha'} W, V \rangle + \langle W, \nabla_{\alpha'} V \rangle$

(compatibility with metric)

(Pf: Ex.)

Remark (1) – (4) defines "Riemannian connection" in Riemannian Geometry.

6.3 Parallel Vector Fields and Holonomy

Def: A vector field $V(t)$ along a curve $\alpha(t)$ is called parallel along α if

$$\boxed{\nabla_{\alpha'} V = 0}.$$

eg: If α is a geodesic, $V = \alpha'(t)$ the tangent vector field along α, then

$$\nabla_{\alpha'} V = \left(\nabla^{\mathbb{R}^3}_{\alpha'} \alpha'\right)_{tan}$$

$$= \alpha''_{tan} = 0$$

$\therefore$ tangent vector field α' is parallel along the geodesic α.

Prop: (Ex 6.3.1 of Oprea)

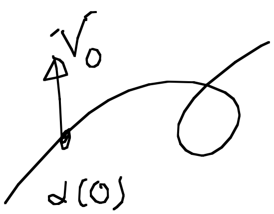
Parallel vector field has constant length.

(Pf: Ex.)

Thm (Existence of parallel transport, Thm 6.3.2 of Oprea)

Let I be an interval containing 0.

$\alpha : I \to M$ be a differentiable curve on M.



Then $\forall\ V_0 \in T_{\alpha(0)}M$, $\exists$ a unique parallel vector field $V(t)$ along α such that $V(0) = V_0$.

Pf: WLOG we assume α is contained in an orthogonal coordinate patch $X(u^1, u^2)$ and $|V_0| = 1$.

Then the parallel vector field $V(t)$ if exists, must also have unit length: $|V(t)| = 1$.

Define as before the moving frame

$$\mathcal{E}_1 = \frac{1}{\sqrt{g_{11}}} \mathbf{X}_1 \quad , \quad \mathcal{E}_2 = \frac{1}{\sqrt{g_{22}}} \mathbf{X}_2$$

on the patch.

Then $V(t) = \cos\theta(t)\, \mathcal{E}_1 + \sin\theta(t)\, \mathcal{E}_2$

for some differentiable function $\theta(t)$ such that

$\theta(t)$ is the angle from $\mathcal{E}_1$ to $V(t)$ at $\alpha(t)$

(not necessary in $(0, 2\pi)$.)

Then $\nabla_{\alpha'} V = 0$

$$\Leftrightarrow \quad \nabla_{\alpha'}\left(\cos\theta\, \mathcal{E}_1 + \sin\theta\, \mathcal{E}_2\right) = 0$$

$$\Leftrightarrow \quad -\sin\theta \cdot \theta'\, \mathcal{E}_1 + \cos\theta\, \nabla_{\alpha'} \mathcal{E}_1 + \cos\theta \cdot \theta'\, \mathcal{E}_2 + \sin\theta\, \nabla_{\alpha'} \mathcal{E}_2 = 0$$

$$\Leftrightarrow \quad -\sin\theta \cdot \theta'\, \mathcal{E}_1 + \cos\theta\,(\omega_{21} \mathcal{E}_2) + \cos\theta \cdot \theta'\, \mathcal{E}_2 + \sin\theta\,(-\omega_{21} \mathcal{E}_1) = 0$$

$$\Leftrightarrow \quad (\theta' + \omega_{21})\big(\underbrace{-\sin\theta\, \xi_1 + \cos\theta\, \xi_2}_{\hookrightarrow \text{ unit length}}\big) = 0$$

$$\Leftrightarrow \quad \theta' + \omega_{21} = 0$$

$$\Leftrightarrow \quad \theta(t) = \theta_0 - \int_0^t \omega_{21}(\alpha(t))\, dt$$

This formula defines the parallel vector field $V = \cos\theta\, \xi_1 + \sin\theta\, \xi_2$ and gives the uniqueness of the theorem. $\#$

__Def__: • If $V(t)$ = parallel vector field along $\alpha : I \to M$ and $0 \in I$,
Then $V(t_1)$ is called the __parallel transport__ of $V(0)$ along α at the point t_1.

• The __angle of rotation__ $-\int_0^t \omega_{21}\, dt$ is called the __holonomy__ along α.

<u>Prop</u> (Ex. 6.3.4 of Oprea) Let $\alpha(t)$ be a curve and

$\theta(t)$ be a diff. function s.t. $\theta(t)$ = angle from $\mathcal{E}_1$ to α'.

Then the geodesic curvature $\boxed{\kappa_g = \dfrac{d\theta}{dt} + \omega_{21}.}$

(Pf: Ex.)

By this proposition, we have

Thm (Thm 6.3.5 of Oprea)

Holonomy is preserved by isometry.

Pf: Since angles and κ_g are preserved. ※

eg (Eg 6.3.8 of Oprea)

Let $M = S^2_R$ parametrized by

$$X(u,v) = (R\cos u\cos v,\ R\sin u\cos v,\ R\sin v). \quad (\cos v \geq 0)$$

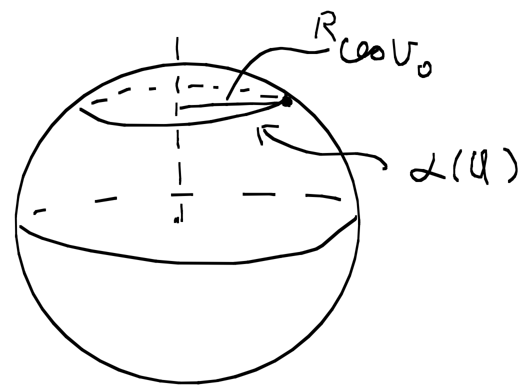
Then

$$\begin{cases} X_1 = (-R\sin u\cos v,\ R\cos u\cos v,\ 0) \\ X_2 = (-R\cos u\sin v,\ -R\sin u\sin v,\ R\cos v) \end{cases}$$

$$\Rightarrow \quad \begin{cases} g_{11} = \langle \mathbf{X}_1, \mathbf{X}_1 \rangle = R^2 \cos^2 v \\ g_{12} = \langle \mathbf{X}_1, \mathbf{X}_2 \rangle = 0 \\ g_{22} = \langle \mathbf{X}_2, \mathbf{X}_2 \rangle = R^2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} E_1 = \dfrac{1}{\sqrt{g_{11}}} \mathbf{X}_1 = \dfrac{1}{R\cos v}(-R\sin u \cos v, R\cos u \cos v, 0) \\ \quad = (-\sin u, \cos u, 0) \\ E_2 = (-\cos u \sin v, -\sin u \sin v, \cos v) \end{cases}$$

Let $\alpha(u)$ = latitude circle at latitude v_0



$$= (R\cos u \cos v_0, R\sin u \cos v_0, R\sin v_0)$$

Then $\alpha'(u) = (-R\sin u \cos v_0, R\cos u \cos v_0, 0)$,

$$\Rightarrow \quad |\alpha'(u)| = R\cos v_0 = \text{constant}.$$

Note that $\alpha(u)$ is just a u-parameter curve of $X(u,v)$,

$$\nabla^{\mathbb{R}^3}_{\alpha'} E_1 = \frac{d}{du} E_1 = (-\cos u, -\sin u, 0)$$

& $$\nabla^{\mathbb{R}^3}_{\alpha'} E_2 = \frac{d}{du} E_2 = (\sin u \sin v_0, -\cos u \sin v_0, 0)$$

$$\Rightarrow \quad \omega_{21} = \langle \nabla_{\alpha'} E_1, E_2 \rangle = \langle \nabla^{\mathbb{R}^3}_{\alpha'} E_1, E_2 \rangle \quad (\text{Since } E_2 \in T_pM)$$

$$= \langle (-\cos u, -\sin u, 0), (-\cos u \sin v_0, -\sin u \sin v_0, \cos v_0) \rangle$$

$$= \sin v_0$$

[One checks

$$\omega_{12} = \langle \nabla^{\mathbb{R}^3}_{\alpha} E_2, E_1 \rangle = \langle (\sin u \sin v_0, -\cos u \sin v_0, 0), (-\sin u, \cos u, 0) \rangle$$

$$= -\sin v_0 = -\omega_{21}$$]

$$\therefore \quad \nabla_{\alpha'} E_1 = \sin v_0 \, E_2 \quad \& \quad \nabla_{\alpha'} E_2 = -\sin v_0 \, E_1 .$$

Note that in this case, $\alpha'(u) = X_1 \,//\, \mathcal{E}_1$.

θ = angle from $\mathcal{E}_1$ to $X_1 \equiv 0$.

$\Rightarrow$ geodesic curvature $\kappa_g = \dfrac{d\theta}{dt} + \omega_{21} = \sin v_0$

$\therefore$ α is a geodesic $\Leftrightarrow$ $v_0 = 0$ (consistent with result given before.)

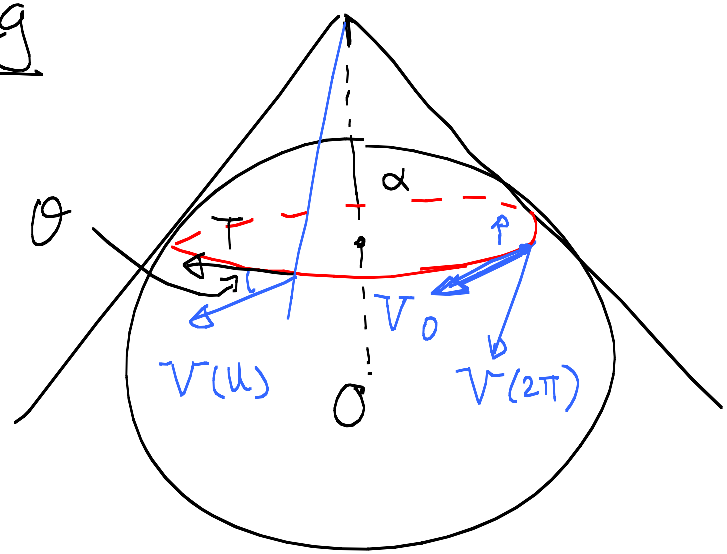
If $V_0 = \alpha'(0) \in T_{\alpha(0)} S^2_R$, then angle from $\mathcal{E}_1$ to $V_0 = \theta(0) = 0$.

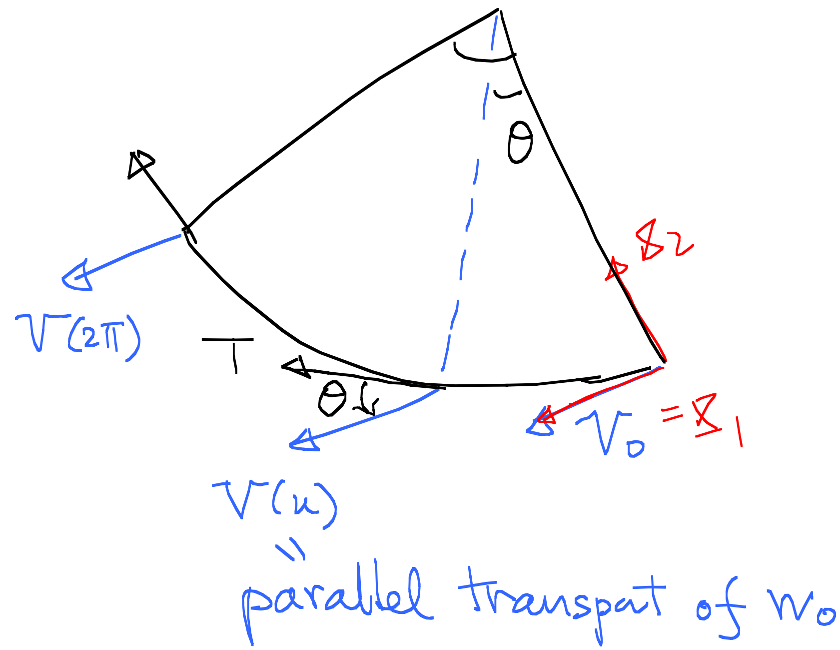
and hence the angle from the <u>parallel transport</u> $V(2\pi)$ along $\alpha(u)$ at $t = 2\pi$ (ie. after 1 complete turn) to $\mathcal{E}_1$ is

$$\theta(2\pi) = \theta(0) - \int_0^{2\pi} \omega_{21}\, dt = -2\pi \sin v_0$$

$\therefore$ in general, parallel transport along a <u>closed</u> curve does <u>not</u> equal to the initial vector.

eg





6.4 Foucault's Pendulem

(Omitted)

6.5 The Angle Excess Theorem

Lets start with a famula for ω_{21}:

Suppose $\mathbf{X}(u^1, u^2)$ = orthogonal coordinate

$$\begin{cases} \mathbf{E}_1 = \dfrac{1}{\sqrt{g_{11}}} \mathbf{X}_1 \\ \mathbf{E}_2 = \dfrac{1}{\sqrt{g_{22}}} \mathbf{X}_2 \end{cases}$$

$\alpha(s)$ = unit speed curve

Then $\alpha'(s) = \sum_i \dfrac{du^i}{ds} \mathbf{X}_i$ and

$$\nabla_{\alpha'(s)} \mathbf{E}_1 = \nabla_{\sum_i \frac{du^i}{ds} \mathbf{X}_i} \left(\frac{1}{\sqrt{g_{11}}} \mathbf{X}_1 \right)$$

$$= \sum_i \frac{du^i}{ds} \nabla_{\mathbf{X}_i} \left(\frac{1}{\sqrt{g_{11}}} \mathbf{X}_1 \right)$$

$$= \sum_i \frac{du^i}{ds}\left[\left(\nabla_{X_i}\frac{1}{\sqrt{g_{11}}}\right)X_1 + \frac{1}{\sqrt{g_{11}}}\nabla_{X_i}X_1\right] \quad \text{(tangential)}$$

$$= \sum_i \frac{du^i}{ds}\left(\nabla_{X_i}\frac{1}{\sqrt{g_{11}}}\right)X_1 + \frac{1}{\sqrt{g_{11}}}\sum_i \frac{du^i}{ds}\left(\sum_j \Gamma^j_{i1} X_j\right)$$

note that $X_1 = \sqrt{g_{11}}\,\mathcal{E}_1 \Rightarrow \langle X_1, \mathcal{E}_2\rangle = 0$.

$$\therefore \quad \omega_{21} = \langle \nabla_{\alpha'(s)}\mathcal{E}_1, \mathcal{E}_2\rangle = \frac{1}{\sqrt{g_{11}}}\sum_i \frac{du^i}{ds}\Gamma^2_{i1}\langle X_2, \mathcal{E}_2\rangle$$

$$= \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\sum_i \Gamma^2_{i1}\frac{du^i}{ds}.$$

For $g_{12}=0$, $\Gamma^2_{11} = -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial u^2}$, $\Gamma^2_{21} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial u^1}$

Hence
$$\omega_{21} = \frac{\sqrt{g_{22}}}{\sqrt{g_{11}}}\left(-\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial u^2}\frac{du^1}{ds} + \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial u^1}\frac{du^2}{ds}\right)$$

$$\Rightarrow \quad \boxed{\omega_{21} = \frac{1}{2\sqrt{g_{11} g_{22}}}\left(-\frac{\partial g_{11}}{\partial u^2}\frac{du^1}{ds} + \frac{\partial g_{22}}{\partial u^1}\frac{du^2}{ds}\right)} \qquad (g_{12}=0)$$

In classical notation $\boxed{\omega_{21} = \frac{1}{2\sqrt{EG}}\left(-E_v \frac{du}{ds} + G_u \frac{dv}{ds}\right)}$ $\quad (F=0)$

Note that the above formula depends only on the curve and the metric of the surface, we've reproved that holonomy is preserved by isometry.

(Omitted, for students interested in Grad school)

Moreover, we see that holonomy are well-defined on abstract surfaces not in $\mathbb{R}^3$.

<u>Remark</u>:

$$\omega_{21} = \frac{1}{2\sqrt{EG}}\left(-E_v \frac{du}{ds} + G_u \frac{dv}{ds}\right)$$

should be considered as a "differential 1-form"

$$\omega_{21} = \frac{1}{2\sqrt{EG}}\left(-E_v\,du + G_u\,dv\right)$$

which gives, for a given tangent vector $\alpha' = u'\mathbf{X}_u + v'\mathbf{X}_v$

the value $\omega_{21}(\alpha') = \frac{1}{2\sqrt{EG}}\left(-E_v\frac{du}{ds} + G_u\frac{dv}{ds}\right)$.

For those who know differential forms, it is easy

to see

$$d\omega_{21} = d\left(-\frac{E_v}{2\sqrt{EG}}\right)\wedge du + d\left(\frac{G_u}{2\sqrt{EG}}\right)\wedge dv$$

$$= \left[\frac{\partial}{\partial u}\left(-\frac{E_v}{2\sqrt{EG}}\right)du + \frac{\partial}{\partial v}\left(-\frac{E_v}{2\sqrt{EG}}\right)dv\right]\wedge du$$

$$+ \left[\frac{\partial}{\partial u}\left(\frac{G_u}{2\sqrt{EG}}\right)du + \frac{\partial}{\partial v}\left(\frac{G_u}{2\sqrt{EG}}\right)dv\right]\wedge dv$$

$$= (\cdot\cdot)\, \cancel{du \wedge du}^{\to 0} - \frac{\partial}{\partial v}\left(\frac{E_v}{2\sqrt{EG}}\right) dv \wedge du$$

$$+ \frac{\partial}{\partial u}\left(\frac{G_u}{2\sqrt{EG}}\right) du \wedge dv + (\quad)\, \cancel{dv \wedge dv}^{\to 0}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right] du \wedge dv$$

using
$dv \wedge du = -du \wedge dv$

$$= -K\sqrt{EG}\, du \wedge dv$$

i.e. $\boxed{d\omega_{21} = -K\, dA}$ where $dA = \sqrt{EG}\, du \wedge dv$

$= \sqrt{\det(g_{ij})}\, du^1 \wedge du^2$

or $\left(d\omega_{12} = K\, dA \right)$ is the volume form

Then Stokes thm $\Rightarrow$ for a simply-connected region R with boundary $\partial R = C$, we have $$\boxed{\iint_R K\,dA = -\iint_R d\omega_{21} = -\oint_{\partial R} \omega_{21}}$$.

$$= -\oint_C \frac{1}{2\sqrt{EG}}\left(-E_v \frac{du}{ds} + G_u \frac{dv}{ds}\right) ds$$

where $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ = tangent vector of the curve C.

i.e. $$\boxed{\iint_R K\,dA = -\oint_C \frac{1}{2\sqrt{EG}}\left(-E_v\, du + G_u\, dv\right)}$$

(Omitted)

Using Green's thm $\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$,

we have $\oint_C \omega_{21} = \oint_C \frac{1}{2\sqrt{EG}}\left(-E_v \frac{du}{ds} + G_u \frac{dv}{ds}\right) ds$

$$= \frac{1}{2}\oint_C \left(-\frac{E_v}{\sqrt{EG}}\right) du + \left(\frac{G_u}{\sqrt{EG}}\right) dv$$

$$= \frac{1}{2}\iint_R \left[\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right] du\,dv$$

$$= \iint_R \frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right] dA$$

$$= -\iint_R K\,dA$$

On the other hand, if $\alpha(t)$ parametrizes the boundary $\partial R = C$,

then $\omega_{21} = \kappa_g - \frac{d\theta}{ds}$, where θ = angle from $\mathcal{E}_1$ to α', and we have

$$\iint_R K\,dA = -\oint_C \left(\kappa_g - \frac{d\theta}{ds}\right) ds = -\oint_C \kappa_g ds + \oint_C d\theta$$

$$= -\oint_C \kappa_g ds + \theta(2\pi) - \theta(0)$$

This gives the

<u>Thm</u> (<u>local Gauss-Bonnet</u>, Thm 6.5.3 of Oprea) $\left(-\int_\beta \omega_{21}\right)$

For a shrinkable simple closed curve β, the (total) holonomy around β may be identified with the total Gaussian curvature evaluated on the (simply-connected) region B inside β.

Furthermore, the total change in θ around β is given by

$$\theta(2\pi) - \theta(0) = \oint_\beta \kappa_g ds + \iint_B K\,dA,$$

where θ = angle from $\mathcal{E}_1$ to β'.

Corollary: If β is a geodesic, then

$$\theta(2\pi) - \theta(0) = \iint_B K .$$

Fact from topology: For shrinkable curve (i.e. boundary of simply-connected region), $\theta(2\pi) - \theta(0) = 2\pi$.

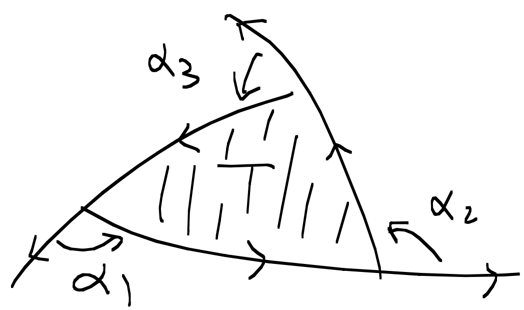
$\therefore$ For simply-connected region B

$$\boxed{2\pi = \oint_{\partial B} k_g \, ds + \iint_B K \, dA} .$$

The above method can be improved to include simply-connected regions with boundary with finitely many non-differentiable points (but still with some condition, see do Carmo.)

In particular, it can be used to study "triangles" with well defined angles at the vertexes (ie non-diff. points):

- A "topological triangles" is a region T topologically equal to a disk with 3 non-differentiable points called vertexes on the boundary and with non-zero <u>external</u> angles α_i, $i=1,2,3$.



<u>Thm</u> (Thm 6.5.6 of Oprea)

For a triangle T with external angles $\alpha_1, \alpha_2, \alpha_3$;

we have

$$\boxed{\sum_j \alpha_j + \int_{\partial T} \kappa_g \, ds + \iint_T K \, dA = 2\pi}$$

Remarks

- Note that the sum of angles compensates the "jump" of the angle function $\theta(t)$ at non-differentiability points of the bdy.

- Let $i_j = \pi - \alpha_j$, $j=1,2,3$, be the interior angles at the vertexes, then

$$\sum_j (\pi - i_j) + \int_{\partial T} k_g\, ds + \iint_T K\, dA = 2\pi$$

$$\Rightarrow \quad 3\pi - (i_1 + i_2 + i_3) + \int_{\partial T} k_g\, ds + \iint_T K\, dA = 2\pi$$

$$\Rightarrow \quad \boxed{\pi - (\hat{i}_1 + \hat{i}_2 + \hat{i}_3) = -\int_{\partial T} k_g\, ds - \iint_T K\, dA}\,.$$

Cor (Cor 6.5.7 of Oprea)

If T is a "geodesic triangle", i.e topologically triangle with geodesic segments as boundary, then

$$\boxed{\pi - (i_1 + i_2 + i_3) = -\iint_T K \, dA}$$

, where i_j = interior angles of T ($j=1,2,3$)

Pf: $k_g = 0$ along ∂T

#

Cor (Cor 6.5.8 of Oprea)

If the surface has constant Gaussian curvature K, then

$$\boxed{\pi - (i_1 + i_2 + i_3) = -K \, \text{Area}(T)}$$

where T = a geodesic triangle

i_j = interior angles of T, $j=1,2,3$.

In particular, if $K \equiv 1$, $(i_1 + i_2 + i_3) - \pi = \text{Area}(T)$ (spherical)

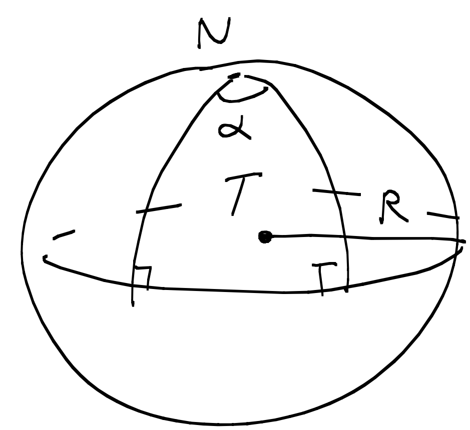
$K \equiv 0$, $i_1 + i_2 + i_3 = \pi$ (Euclidean)

$K \equiv -1$, $\pi - (i_1 + i_2 + i_3) = \text{Area}(T)$ (hyperbolic)

(These are the classical <u>angle excess theorems</u> for non-Euclidean geometry.)

<u>eg</u>: On S^2_R, $K \equiv \frac{1}{R^2}$, Sum of interior angles of an geodesic triangle on S^2_R $= \pi + \frac{1}{R^2}\,\text{Area}$.

For the triangle T in the figure



$$\alpha + \frac{\pi}{2} + \frac{\pi}{2} = \pi + \frac{1}{R^2}\,\text{Area}(T)$$

$$\Rightarrow \boxed{\text{Area}(T) = R^2 \cdot \alpha}$$

If $\alpha = 2\pi$, then $\text{Area}(\text{hemisphere of radius } R) = 2\pi R^2$

6.6 The Gauss-Bonnet Theorem

Def (Triangulation)

A triangulation of a surface M is a finite family $\mathcal{T}$ of triangles $\mathcal{T} = \{T_1, \cdots, T_n\}$, T_i are topological triangles

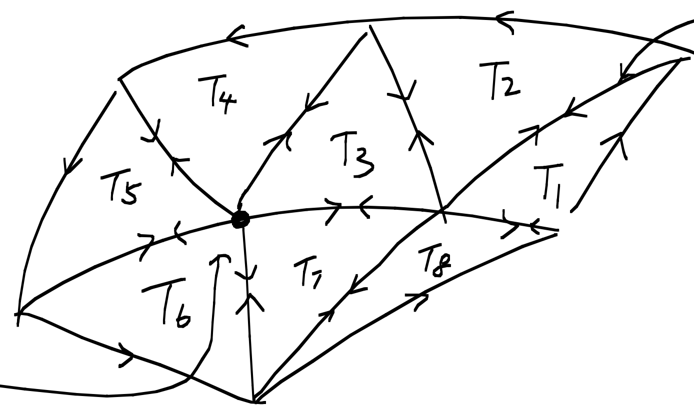
s.t. (1) $\bigcup_{i=1}^{n} T_i = M$

(2) If $T_i \cap T_j \neq \phi$, then

$$T_i \cap T_j = \begin{cases} \text{common edge of } T_i \,\&\, T_j, \text{ or} \\ \text{common vertex of } T_i \,\&\, T_j. \end{cases}$$

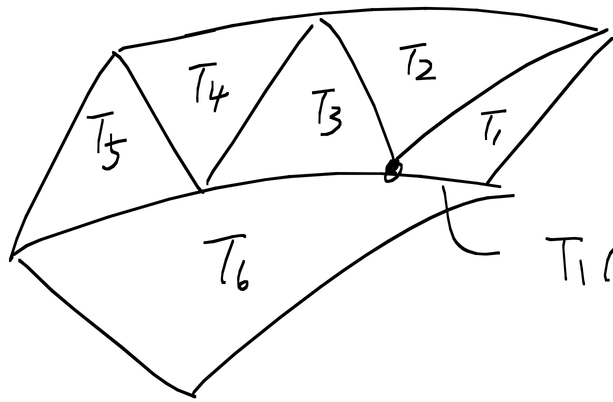
eg

$T_1 \cap T_2$



$T_3 \cap T_6$ = vertex

a triangulation



Not a triangulation

$T_1 \cap T_6 \neq$ edge of T_6

Def : Given a triangulation $\mathcal{T}$ of a surface M we denote

$$\begin{cases} F = \text{number of triangles (faces)} \\ E = \text{number of sides (edges)} \\ V = \text{number of vertices} \end{cases}$$

Then the number $\boxed{\chi = F - E + V}$

is called the Euler-Poincaré characteristic of the triangulation.

Prop: Every (regular) surface admits a triangulation.

Prop : Let
- M = <u>oriented</u> (regular) surface
- $\{\mathbf{X}_\alpha : U_\alpha \to M\}_{\alpha \in A}$ = family of coordinate patches <u>compatible</u> with the <u>orientation</u> of M.

Then $\exists$ a triangulation $\mathcal{T}$ of M s.t.

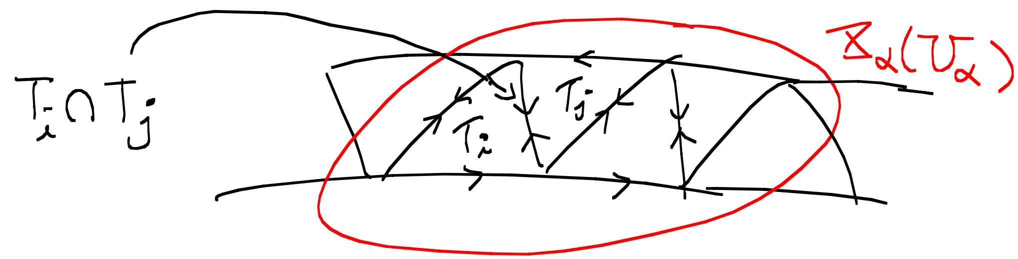
(1) $\forall\, T \in \mathcal{T}$, $\exists\, \alpha \in A$ s.t.

$$\boxed{T \subset \mathbf{X}_\alpha(U_\alpha)}$$

(2) If $\forall\, T \in \mathcal{T}$, ∂T is positively oriented.

Then $\forall\, T_i \cap T_j$ = common edge of T_i & T_j, and

$$\boxed{T_i \text{ \& } T_j \text{ determine } \underline{\text{opposite}} \text{ orientation in } T_i \cap T_j.}$$



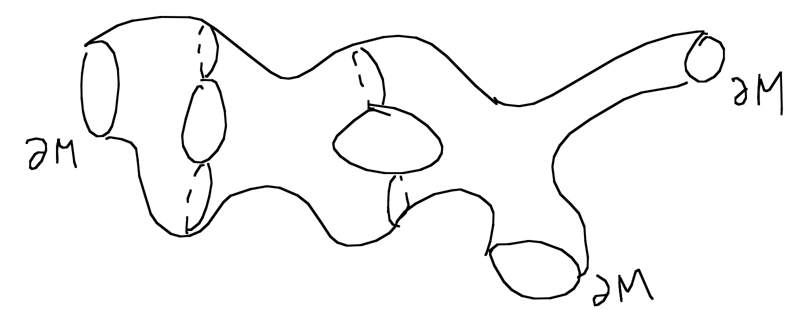
Prop: $\chi(M)$ is a topological invariant of the surface M.

Now we are ready to prove

Thm (Gauss–Bonnet Theorem, Thm 6.6.1 of Oprea)
(see do Carmo for a more detailed version)

If M is a compact oriented surface with boundary ∂M made up of a finite number of smooth closed curves, then

$$\boxed{\int_{\partial M} \kappa_g \, ds + \iint_M K \, dA = 2\pi \chi(M).}$$



Pf: Consider a triangulation $\mathcal{T}$ of M s.t. $\forall\ T_j \in \mathcal{T}$, T_j is contained in a coordinate patch of a family of orthogonal coordinate patch compatible with the orientation of M.

Furthermore, ∂T_j are oriented positively. Then we obtain opposite orientations in common edges.

Applying local Gauss-Bonnet theorem to $T_j \in \mathcal{T}$, we have (θ_{jk} = external angles of T_j)

$$\sum_{k=1}^{3} \theta_{jk} + \int_{\partial T_j} k_g(s)\,ds + \iint_{T_j} K\,dA = 2\pi .$$

Summing up all T_j, $j=1,\cdots,F \leftarrow$ no of faces.,

we have

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk}+\sum_{j=1}^{F}\int_{\partial T_j}\kappa_g\,ds+\sum_{j=1}^{F}\iint_{T_j}K\,dA=2\pi F$$

Note that $$\int_{\partial T_j}\kappa_g\,ds=\sum_{k=1}^{3}\int_{(\partial T_j)_k}\kappa_g\,ds$$ $\leftarrow$ the sides of T_j

& on a common edge

$$\partial T_i\cap\partial T_j=(\partial T_j)_k=(\partial T_i)_l$$ with opposite orientation

we have $$\int_{(\partial T_j)_k}\kappa_g\,ds=-\int_{(\partial T_i)_l}\kappa_g\,ds$$

Therefore, all the integrals of interior edges sum to <u>zero</u>

$$\Rightarrow \quad \sum_{j=1}^{F} \int_{\partial T_j} \kappa_g \, ds = \int_{\partial M} \kappa_g \, ds$$

Note also that

$$\sum_{j=1}^{F} \iint_{T_j} K \, dA = \iint_{M} K \, dA ,$$

we have

$$\boxed{\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{jk} + \int_{\partial M} \kappa_g \, ds + \iint_{M} K \, dA = 2\pi F} \quad \text{— (1)}$$

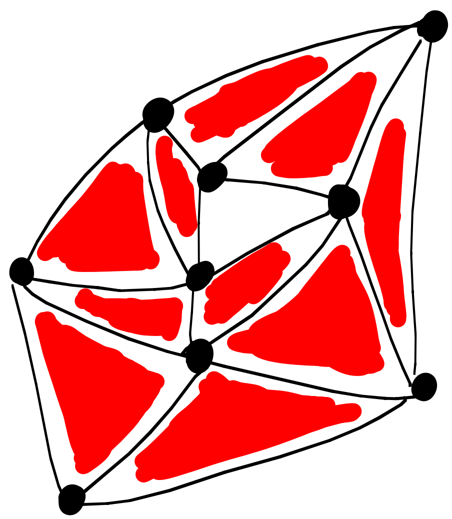
Let $\varphi_{jk} = \pi - \theta_{jk}$ be the interior angles of T_j

Then

$$\boxed{\sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{jk} = 3\pi F - \sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk}} \quad \text{— (2)}$$

Let
$$\begin{cases} E_e = \text{number of } \underline{\text{external}} \ \underline{\text{edges}} \text{ of } J \\ E_i = \text{number of } \underline{\text{internal}} \ \underline{\text{edges}} \text{ of } J \\ V_e = \text{number of } \underline{\text{external}} \ \underline{\text{vertices}} \text{ of } J \\ V_i = \text{number of } \underline{\text{internal}} \text{ vertices of } J \end{cases}$$

<u>eg</u>:



$E_e = 8$

$E_i = 11$

$V_e = 8$

$V_i = 1$

$F = 10$

$E = 19 = E_e + E_i$

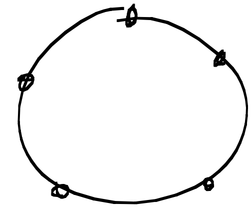
$V = 9$

[This figure show a general case. In our case, the 2 boundary curves should be smooth without corner.]

Now (i) ∂M closed $\Rightarrow$ $\boxed{E_e = V_e}$

Since each boundary curve $C_i \sim S^1$

$\Rightarrow$ # edges on C_i = # vertices on C_i.



(ii) $\boxed{3F = 2E_i + E_e}$

Pf: $3F$ = # of edges with overlap; internal edges counted exactly twice, external edges counted exactly once. #

In fact, if $F=1$, then $E_i = 0$, $E_e = 3$

$\therefore$ $3F = 2E_i + E_e$.

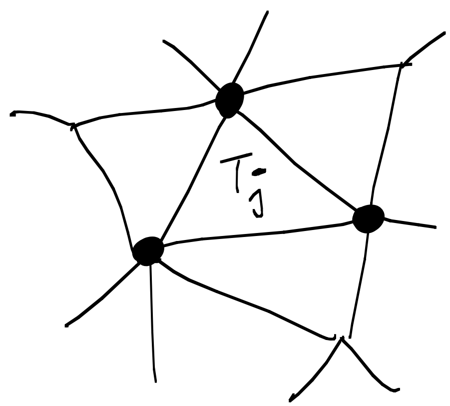
Suppose the formula is true for $F = k$.

Then for $F = k+1$, we consider 3 cases:

(a) $\exists$ a T_j such that all sides of ∂T_j are internal.

Then removing T_j, we obtained a new triangulation (on a new surface) with

$$\tilde{F} = F-1, \quad \tilde{E}_e = E_e + 3, \quad \tilde{E}_i = E_i - 3$$



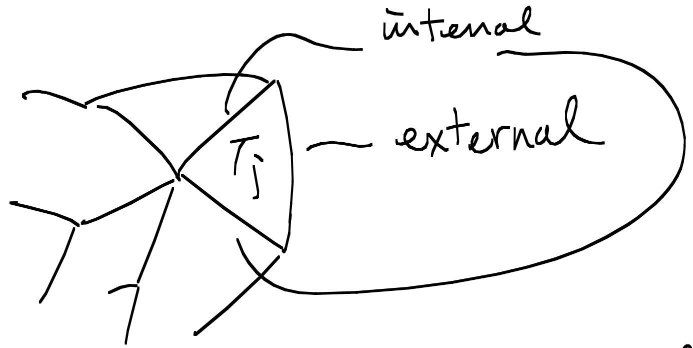
Induction hypothesis

$$\Rightarrow \quad 3\tilde{F} = 2\tilde{E}_i + \tilde{E}_e$$

$$\Rightarrow \quad 3(F-1) = 2(E_i - 3) + (E_e + 3)$$

$$\Rightarrow \quad 3F = 2E_i + E_e$$

(b) $\exists\ T_j \in \mathcal{T}$ s.t. 2-sides of T_j are internal.



Then removing T_j, we have a new triangulation with

$$\tilde{F} = F-1, \quad \tilde{E}_e = E_e - 1 + 2 = E_e + 1, \quad \& \quad \tilde{E}_i = E_i - 2$$

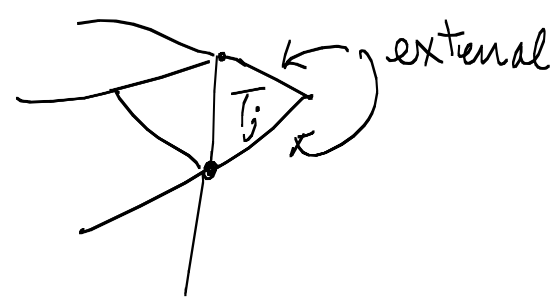
Induction hypo. $\Rightarrow \quad 3\tilde{F} = 2\tilde{E}_i + \tilde{E}_e$

$$\Rightarrow \quad 3(F-1) = 2(E_i - 2) + (E_e + 1)$$

$$\Rightarrow \quad 3F = 2E_i + E_e.$$

Finally (c) $\exists\ T_j \in \mathcal{T}$ with

1 internal side



Then removing T_j, we obtained a new triangulation with

$$\tilde{F} = F-1, \quad \tilde{E}_e = E_e - 2 + 1 = E_e - 1, \quad \& \quad \tilde{E}_i = E_i - 1.$$

Induction hypo $\Rightarrow$ $3\tilde{F} = 2\tilde{E}_i + \tilde{E}_e$

$\Rightarrow$ $3(F-1) = 2(E_i - 1) + (E_e - 1)$

$\Rightarrow$ $3F = 2E_i + E_e$

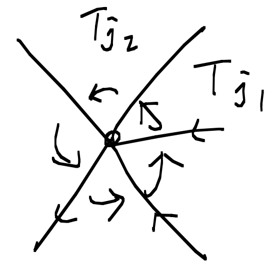
Altogether, induction $\Rightarrow$ $3F = 2E_i + E_e$ is true for all $\mathcal{J}$. ※

Applying (ii) to (2), we have

$$\boxed{\begin{aligned} \sum_{j=1}^{F} \sum_{k=1}^{3} \theta_{jk} &= 3\pi F - \sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk} \\ &= 2\pi E_i + \pi E_e - \sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk} \end{aligned}} \quad \text{--- (3)}$$

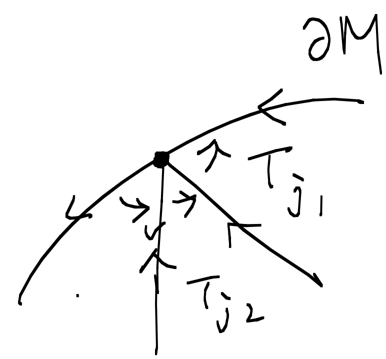
To calculate $\sum_{j=1}^{F} \sum_{k=1}^{3} \varphi_{jk}$, we observe the following 2 cases:

(a) <u>internal vertices</u>:



$$\boxed{\sum_{\substack{\text{internal} \\ \text{vertices}}} \text{interior angles} = 2\pi V_i}$$

(b) <u>external vertices</u> on ∂M (assumed to be smooth)



$$\boxed{\sum_{\substack{\text{ext vertices} \\ \text{on } \partial M_{(\text{smooth})}}} \left(\text{interior angles} \right) = \pi V_e}$$

Therefore,

$$\boxed{\sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk} = 2\pi V_i + \pi V_e}$$

And hence (3) $\Rightarrow$

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j=1}^{F}\sum_{k=1}^{3}\varphi_{jk}$$

$$= 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_e$$

$$= 2\pi(E_i - V_i)$$

since $E_e = V_e$ used twice

$$= 2\pi(E_i - V_i + E_e - V_e)$$

$$= 2\pi(E - V)$$

Sub. into the famula (1):

$$\sum_{j=1}^{F}\sum_{k=1}^{3}\theta_{jk} + \int_{\partial M} k_g\,ds + \iint_M K\,dA = 2\pi F$$

we have

$$2\pi(E-V) + \int_{\partial M} k_g\, ds + \iint_M K\, dA = 2\pi F$$

$$\Rightarrow \quad \int_{\partial M} k_g\, ds + \iint_M K\, dA = 2\pi(F-E+V) = 2\pi\chi(M).$$

#

<u>Remark</u>: If ∂M has finite many vertexes (non-diff. points) with external angles $\theta_1, \cdots, \theta_p$. Then the above proof can be modified to get

$$\boxed{\sum_{i=1}^{p}\theta_i + \int_{\partial M} k_g\, ds + \iint_M K\, dA = 2\pi\chi(M)}$$

See doCarmo for the details

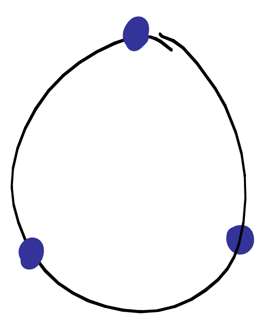
Cor (Cor 6.6.2 of Oprea)

If M is a compact oriented surface without boundary, then

$$\boxed{\iint_M K \, dA = 2\pi \chi(M).}$$

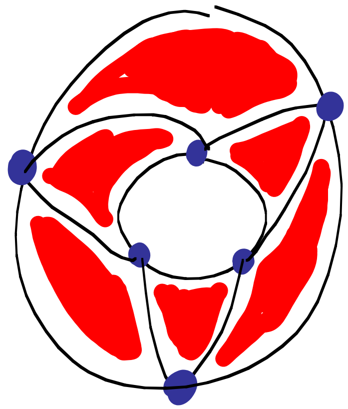
egs: Euler-Poincaré Characteristic

① Disk



$$\left.\begin{array}{l} F = 1 \\ E = 3 \\ V = 3 \end{array}\right\} \Rightarrow \chi = 1 - 3 + 3 = 1$$

(2) Cylinder, annulus



$$\left.\begin{aligned} F &= 6 \\ E &= 12 \\ V &= 6 \end{aligned}\right\} \Rightarrow \chi = 6 - 12 + 6 = 0$$

Thm (Classification of Surfaces)

(without boundary)

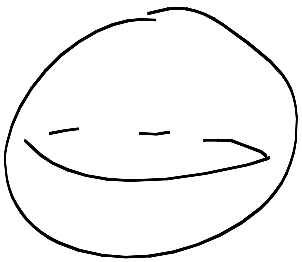
Let M = compact, oriented, connected regular surface;

(Note: This includes all compact connected surfaces in $\mathbb{R}^3$)

(oriented)

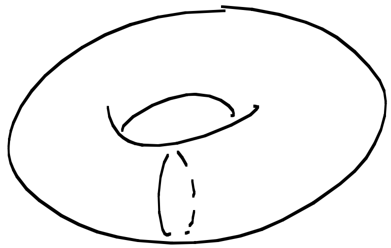
Then one of the values $2, 0, -2, \dots, -2n, \dots$ is assumed by the Euler-Poincaré characteristic $\chi(M)$.

Furthermore, if $M' =$ cpt, oriented, connected regular (without boundary) surface s.t. $\chi(M') = \chi(M)$. Then M' is homeomorphic to M.



$\chi = 2$

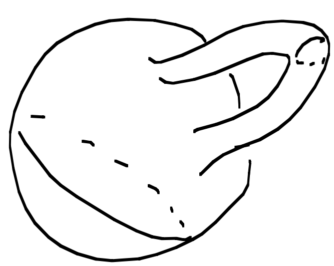
sphere



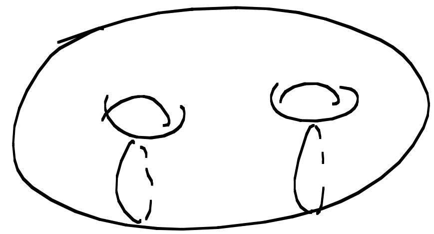
$\chi = 0$

torus

$\cong$



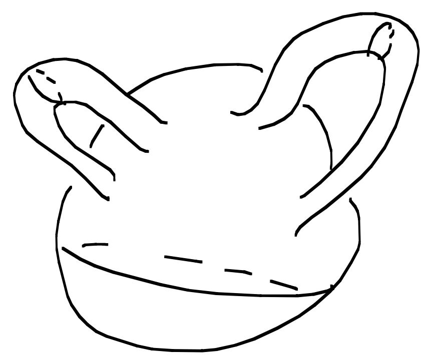
sphere with 1 handle



$\chi = -2$

2-torus

$\cong$



sphere with 2 handles

. . .

. . .

Remark: The number of handles added to the sphere is called the genus g of the resulting surface.

And we have

$$\boxed{g(M) = \frac{2-\chi(M)}{2}}$$

i.e.

$$\boxed{\chi = 2-2g}$$

6.7 Applications of Gauss-Bonnet

Application 1

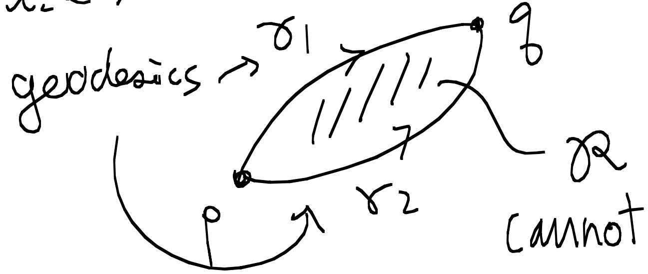
eg (The hyperboloid of one sheet, eg 6.7.1 of Oprea)

We'll prove a more general result given in do Carmo.

Lemma Let M = orientable surface with $K_M \leq 0$.

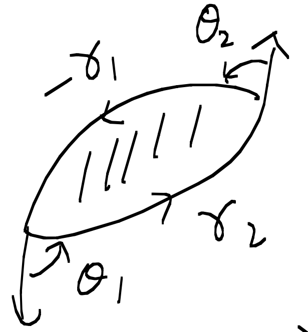
Then 2 geodesics γ_1 & γ_2 starting from a point $p \in M$ cannot meet again at a point $q \in M$ such that γ_1 & γ_2 constitute the boundary of a simple region $\mathcal{R}$ of M.

[simple region = compact region homeomorphic to a disk with boundary = simple closed piecewise diff. curves] i.e.



cannot be simple!

Pf: Assume not, we have



$$\iint_R K + \theta_1 + \theta_2 = 2\pi \qquad \left(\int_{\gamma_i} k_g = 0 \right)$$

$$\Rightarrow \quad \theta_1 + \theta_2 \geq 2\pi \qquad (K_M \leq 0)$$

However $\gamma_1 \neq \gamma_2 \Rightarrow |\theta_i| < \pi$

Otherwise, γ_1, γ_2 have same initial point and initial tangent vector at the end points when parametrized by arc-length. Uniqueness Thm on geodesic $\Longrightarrow \gamma_1 = \gamma_2$.

We have a contradiction. $\times\!\!\times$

Lemma Let M = surface homeo to a cylinder

$K_M < 0$.

Then M has at most one simple closed geodesic.

(geodesic $\Rightarrow$ smooth)

Pf: By assumption, $\exists$ homeomorphism

$$\phi : M \to \mathbb{R}^2 \setminus \{0\}.$$

Let $\Gamma \subset M$ be a simple closed geodesic,

then $\phi(\Gamma)$ is a simple closed curve in $\mathbb{R}^2 \setminus \{0\}$.

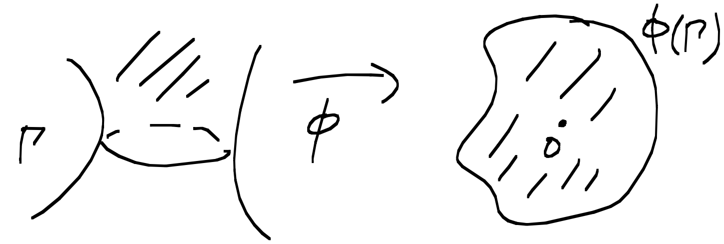
Jordan Thm $\Longrightarrow$ $\phi(\Gamma)$ is a boundary of a region Ω

homeomorphic to a disk in $\mathbb{R}^2$ (not $\mathbb{R}^2 \setminus \{0\}$ in general).

If $0 \notin \Omega$, then $\phi^{-1}(\Omega)$ is a simple region of M

with boundary $\partial(\phi^{-1}(\Omega)) = \Gamma$. By Gauss-Bonnet $\left(\begin{array}{l}\text{previous lemma} \\ \text{with } \theta_1 = \theta_2 = 0\end{array}\right)$

$$0 > \iint_{\phi^{-1}(\Omega)} K = 2\pi$$ which is a contradiction.

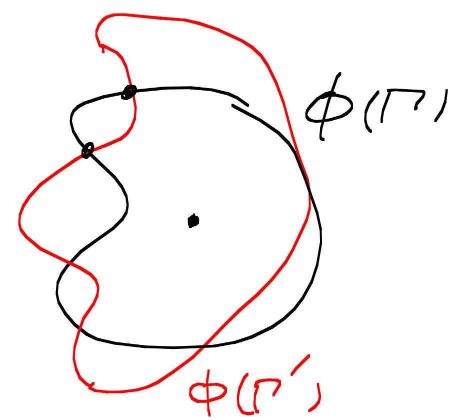


$\therefore$ $0 \in \Omega$.

Suppose Γ' = another simple closed geodesic in M.

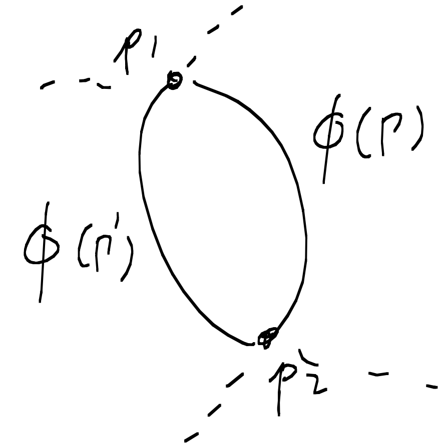
<u>Claim</u>: $\Gamma' \cap \Gamma$ = empty set

Pf: If not, then $\phi(\Gamma') \cap \phi(\Gamma)$ non empty.



If $\phi(\Gamma') \cap \phi(\Gamma)$ contains at least 2 points.

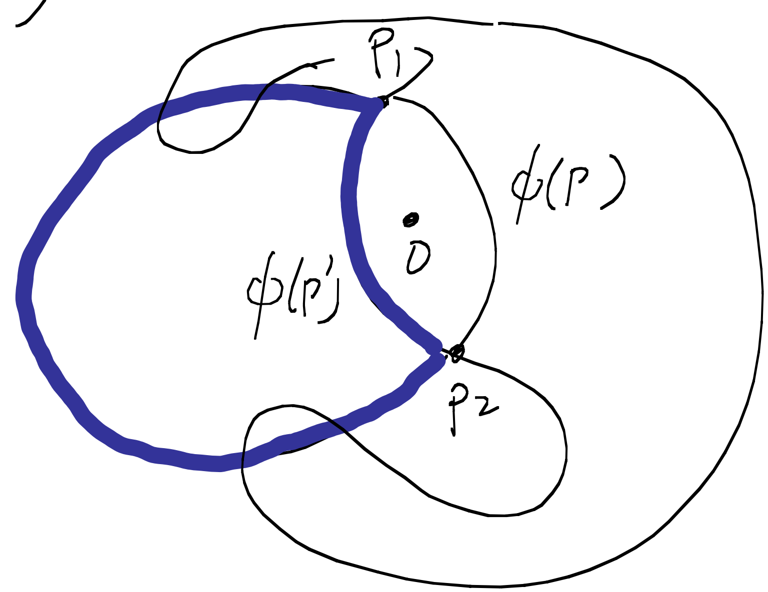
Then one can take 2 "consecutive" intersection points p_1, p_2 such that one of the arc between p_1, p_2 of $\phi(\Gamma)$ and one of the arc between p_1, p_2 of $\phi(\Gamma')$ has no other intersection. Then



previous lemma $\Longrightarrow$

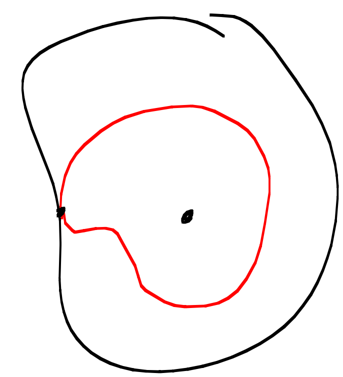
$0 \in$ region bounded by this 2 arcs.

(Otherwise, we obtain a situation contradicting the previous lemma.)



But, then the arc of $\phi(\Gamma')$ together with the complement of the arc in $\phi(\Gamma)$ will bound a domain homeomorphic to the disk; contradicting the previous lemma again.

Next, if $\phi(\Gamma) \cap \phi(\Gamma')$ contains only 1 point. Then $\phi(\Gamma)$ enclosed by $\phi(\Gamma')$ or vice versa.

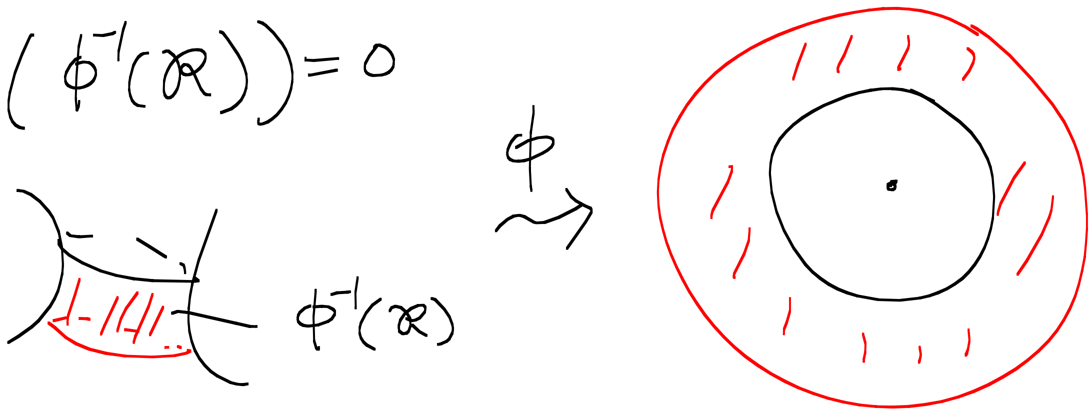


$\therefore$ $\phi(\Gamma)$ and $\phi(\Gamma')$ bounds a simple region again which is impossible.

This proves the claim.

Finally, by the claim, $\phi(\Gamma) \cap \phi(\Gamma')$ empty & hence $\phi(\Gamma)$ & $\phi(\Gamma')$ bounds a domain $\mathcal{R}$ homeomorphic to a <u>cylinder</u>. $\Longrightarrow$ $\chi(\mathcal{R}) = 0$.

$$\Rightarrow \quad \chi(\phi^{-1}(\mathcal{R})) = 0$$



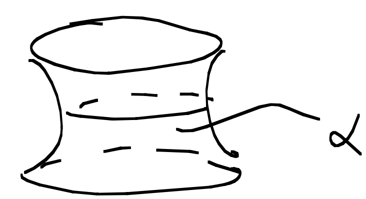
Gauss-Bonnet

$$\Rightarrow \quad \iint_{\phi^{-1}(\mathcal{R})} K + \sum_{i=1}^{2} \int_{\Gamma_i} k_g = 2\pi \chi(\phi^{-1}(\mathcal{R}))$$

$$\int_{\Gamma_i} k_g = 0, \qquad 2\pi\chi(\phi^{-1}(\mathcal{R})) = 0$$

$$\Rightarrow \quad \iint_{\phi^{-1}(\mathcal{R})} K = 0$$ contradicting the assumption that $K<0$. #

For the hyperboloid of one sheet $M : x^2+y^2-z^2=1$. M is clearly homeomorphic to a cylinder and $K_M<0$. The central circle $\alpha(v) = (\cos v, \sin v, 0)$



is a simple closed geodesic. Then by the Lemma, it is the only simple closed geodesic on M.

Reading exercise: Read the remaining of the § 6.7 of the Oprea related to the applications of Gauss-Bonnet theorem to the Jacobi's Bisection Theorem and Hadamard's Theorem.

Ch1 Curve:

- arc length parametrization $\Leftrightarrow$ $|\alpha'(s)| = 1$
- Frenet frame: "Basis of $\mathbb{R}^3$" adapt to & moving along the curve

  Frenet famula
- curvature and torsion, implications of curvature & torsion

Ch2 Surfaces in $\mathbb{R}^3$:

- Regular surface, tangent plane, normal vector, Gauss map
- Shape operator ~ rate of change of normal (up to sign)

  ~ rate of change of tangent plane

  ~ differential of the Gauss map
- normal curvatures

  principal curvatures ~ eigenvalues of the shape operator

  principal directions ~ eigendirections of the shape operator

Ch3 Curvatures:

- Gauss curvature, mean curvature & their calculation

- compatibility condition: Gauss equation & Codazzi-Mainardi eqt.
  $\Gamma_{ij}^k, \dots$
- Effects of curvatures

<u>Ch4</u> Constant Mean Curvature Surfaces:

- minimal surfaces $\leftrightarrow$ area minimizing
- constant mean curvature surfaces $\leftrightarrow$ area minimizing with volume constraints
- Ros' theorem, Alexandrov's theorem

<u>Ch5</u> Geodesics, Metrics & Isometries

- Geodesic ~ "straightest" ~ tangent vector doesn't change wrt the surface.
  $$\Leftrightarrow \nabla_{\alpha'}\alpha' = \left(\nabla^{\mathbb{R}^3}_{\alpha'}\alpha'\right)_{tan} = 0$$
- Geodesic equation, geodesic curvature
- Special case: surface of revolution (Clairaut patch)
  Clairaut relation

- metrics $\sim$ " $ds^2 = g_{ij}\,dx^i dx^j$ " $\sim \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$

  = the first fundamental form

- Isometry = metrics preserving diffeomorphism $\left(f^* ds_N^2 = ds_M^2 \right)$
- Conformal = angle preserving $\left(f^* ds_N^2 = \lambda^2 ds_M^2 \right)$

  (pull-back metric $f^* ds_N^2 = g_{ij}(f)\,df^i df^j$

  $= \sum g_{ij}(f) \frac{\partial f^i}{\partial x^k} \frac{\partial f^j}{\partial x^l}\,dx^k dx^l$)

<u>Ch6</u> Holonomy & the Gauss-Bonnet Thm

- Holonomy : $\{E_1, E_2\}$ orthonormal frame field along α

$$\Rightarrow \begin{cases} \nabla_{\alpha'} E_1 = \omega_{21} E_2 \\ \nabla_{\alpha'} E_2 = -\omega_{21} E_1 \end{cases}$$

$$\Rightarrow \quad \theta(t) - \theta(0) = -\int_0^t \omega_{21}\,dt = \text{holonomy along } \alpha$$

where $\theta = \measuredangle(E_1, \alpha')$

Note: $\boxed{\kappa_g = \frac{d\theta}{dt} + \omega_{21}}$ & $\boxed{\oint_{\partial R} \omega_{21} = -\int_R K\,dA}$ $\left(\boxed{d\omega_{21} = -K\,dA}\right)$

• Gauss-Bonnet Thm

$$\underbrace{\sum \theta_i}_{\text{exterior angles}} + \int_{\partial M} \kappa_g + \int_M K = 2\pi \underbrace{\chi(M)}_{\text{Euler-Poincaré characteristic}}$$

($\Rightarrow$ Classical angle excess theorem)

[• Geodesic Polar coordinates, Gauss-lemma, surfaces of constant curvature, conjugate point] $\longrightarrow$ (not included in final exam)

(End)

Final Exam: • 4 questions, answer all.

• covers all materials in the notes, HW assignments, reading exercises, exercises marked in the notes, and any thing that can be derived from the above materials in a reasonable manners.

6.8 Geodesic Polar Coordinates

Let $p \in M$,

$\{e_1, e_2\}$ orthonormal basis of T_pM.

For any unit tangent vector $W = \cos v\, e_1 + \sin v\, e_2 \in T_pM$,

we denote by $\alpha_W(u)$ the unique unit speed geodesic

with

$$\begin{cases} \alpha_W(0) = p \\ \alpha'_W(0) = W \end{cases}$$

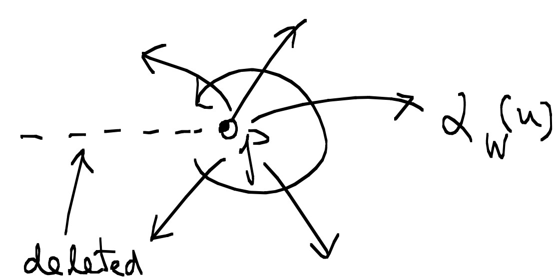
(i.e. $u =$ arc-length parameter of α from $p = \alpha(0)$)

Then one can show that

$$\mathbf{X}(u, v) = \alpha_{(\cos v\, e_1 + \sin v\, e_2)}(u)$$

defines a coordinate patch "around" p.

not exactly:



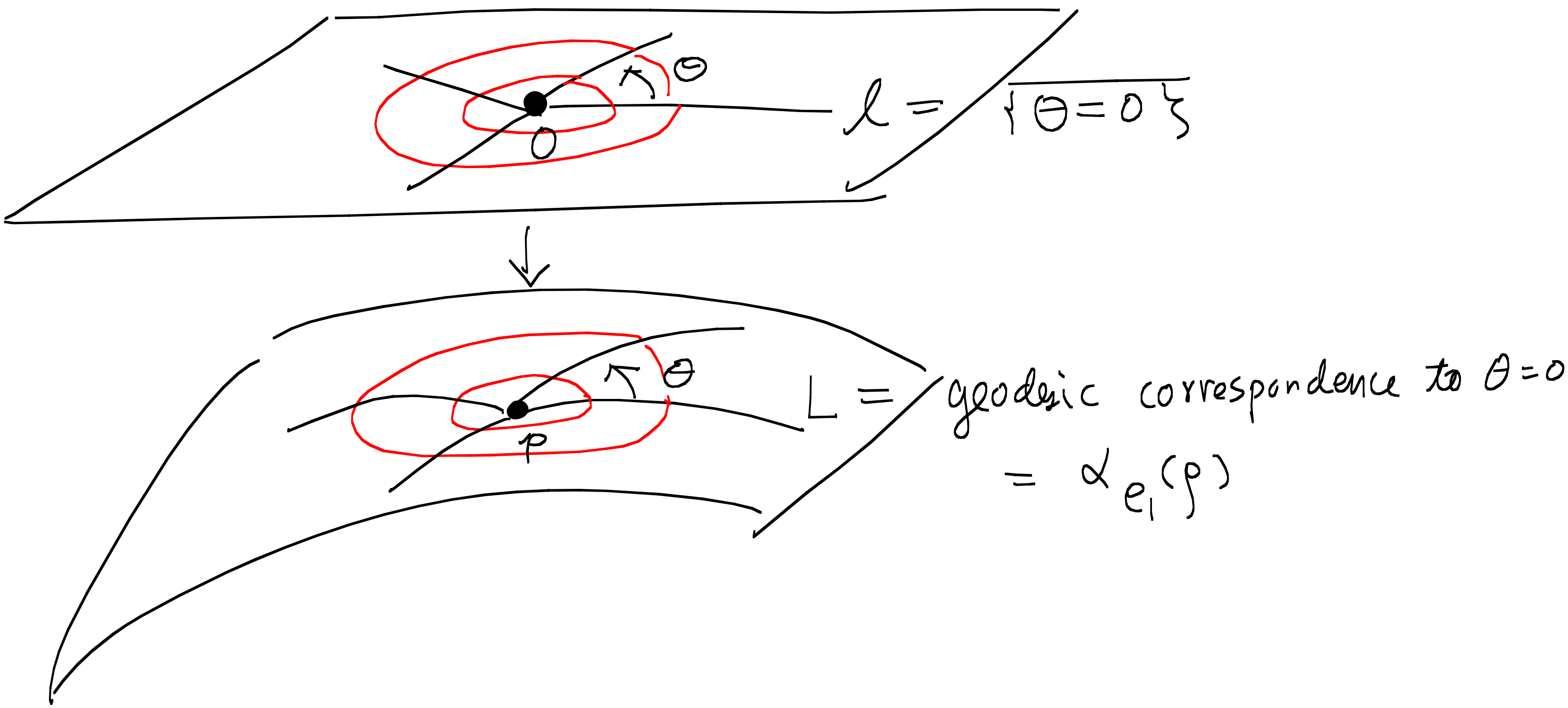
[$\mathbf{x}$ may define only for u up to a finite number.]

(Thm 6.8.1 of Oprea. See §4.6 of do Carmo for rigorous discussion.)

This coordinate patch is called a <u>geodesic polar coordinates</u>

It is in fact more convenient to write (ρ, θ) for the parameters (u, v)

<u>eg</u>: If $M = \mathbb{R}^2 \subset \mathbb{R}^3$, then (ρ, θ) are just the usual polar coordinates on the Euclidean space.



- the θ-parameter curve $= \mathbb{X}(\rho_0, \theta)$ is called a geodesic circles with radius ρ_0;
- the ρ-parameter curve $= \mathbb{X}(\rho, \theta_0)$ is called a radial geodesic (defined up to certain $\rho(\theta_0)$).

Gauss Lemma: The family of geodesic circles is orthogonal to the family of radial geodesics.

Pf: We use the fact the $\theta =$ constant is a geodesics parametrized by the arc-length ρ,

i.e. $\rho = t, \ \theta = \text{const.}$.

By the equations of geodesic:

$$\begin{cases} \dfrac{d^2\rho}{dt^2} + \Gamma^1_{11}\left(\dfrac{d\rho}{dt}\right)^2 + 2\Gamma^1_{12}\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \Gamma^1_{22}\left(\dfrac{d\theta}{dt}\right)^2 = 0 \\ \dfrac{d^2\theta}{dt^2} + \Gamma^2_{11}\left(\dfrac{d\rho}{dt}\right)^2 + 2\Gamma^2_{12}\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \Gamma^2_{22}\left(\dfrac{d\theta}{dt}\right)^2 = 0 \end{cases}$$

$$\Rightarrow \Gamma^1_{11} = \Gamma^2_{11} = 0,$$

i.e.
$$\begin{cases} \displaystyle\sum_{\ell=1}^{2} g^{2\ell}\left(\frac{\partial g_{\ell 1}}{\partial u^1}+\frac{\partial g_{1\ell}}{\partial u^1}-\frac{\partial g_{11}}{\partial u^\ell}\right)=0 \\ \displaystyle\sum_{\ell=1}^{2} g^{1\ell}\left(\frac{\partial g_{\ell 1}}{\partial u^1}+\frac{\partial g_{1\ell}}{\partial u^1}-\frac{\partial g_{11}}{\partial u^\ell}\right)=0, \end{cases}$$

i.e.
$$(*)\quad \begin{cases} g^{21}\dfrac{\partial g_{11}}{\partial \rho}+g^{22}\left(2\dfrac{\partial g_{12}}{\partial \rho}-\dfrac{\partial g_{11}}{\partial \theta}\right)=0 \\ g^{11}\dfrac{\partial g_{11}}{\partial \rho}+g^{12}\left(2\dfrac{\partial g_{12}}{\partial \rho}-\dfrac{\partial g_{11}}{\partial \theta}\right)=0 \end{cases}$$

To find $\dfrac{\partial g_{12}}{\partial \rho}$, we observe that

$$g_{11}=\langle X_1, X_1\rangle = |X_1|^2$$

By construction $X(\rho,\theta)=\alpha_{w_\theta}(\rho)$ where $w_\theta=\cos\theta\, e_1+\sin\theta\, e_2$

$$\Rightarrow \quad |X_1| = |\alpha'_{w_\theta}(\rho)| = |w_\theta| = 1$$

$$\therefore \quad \boxed{g_{11}\equiv 1}.$$

Putting back into $(*)$, we have

$$g^{22} \frac{\partial g_{12}}{\partial \rho} = g^{12} \frac{\partial g_{12}}{\partial \rho} = 0$$

Since $(g^{ij}) = (g_{ij})^{-1}$, g^{22}, g^{12} cannot vanish simultaneously.

$$\Rightarrow \boxed{\frac{\partial g_{12}}{\partial \rho} = 0}.$$

By construction,

$$g_{12} = \langle X_1, X_2 \rangle$$

where

$$X_2 = \frac{d}{d\theta} \left[\alpha w_\theta(\rho) \right].$$

Since $\alpha w_\theta(0) =$ center point p, we have

$$\lim_{\rho \to 0} |X_2| = 0.$$

$\therefore \quad |g_{12}| \le |X_1||X_2| = |X_2| \to 0 \quad$ as $\rho \to 0$

Since $\dfrac{\partial g_{12}}{\partial \rho} = 0$, we have $\boxed{g_{12} = 0}$. ##

<u>Remark</u>: We have proved $g_{11} = 1$, $g_{12} = 0$. $g_{22} > 0$ is clear from $g_{11}g_{22} - g_{12}^2 > 0$. This proves Gauss lemma 6.8.5 of Oprea. However, one in fact has a stronger version:

<u>Thm</u> (Gauss Lemma) For geodesic polar coordinates (ρ, θ), the 1^{st} fundamental form $(g_{ij}(\rho, \theta))$ satisfies

$$g_{11} = 1, \quad g_{12} = 0, \quad \text{and}$$

$$\lim_{\rho \to 0} g_{22} = 0, \quad \lim_{\rho \to 0} \frac{\partial}{\partial \rho}\left(\sqrt{g_{22}}\right) = 1.$$

(In classical notations: $E=1$, $F=0$, $\lim_{\rho\to 0} G=0$, $\lim_{\rho\to 0}(\sqrt{G})_\rho = 1$)

<u>eg</u>: Euclidean space: $dx^2+dy^2 = dr^2+r^2 d\theta^2$

(i.e. $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in rectangular coordinates

& $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ in polar coordinates)

$E=1$, $F=0$, $G=r^2$

$\lim_{r\to 0} G = 0$, $\lim_{r\to 0}(\sqrt{G})_r = \lim_{r\to 0}\frac{\partial r}{\partial r} = 1$.

<u>Pf</u> of <u>the Gauss lemma (strong version)</u>:

We've proved $g_{11}=1$, $g_{12}=0$. To check the last 2 limits, we use the <u>normal</u> coordinates (u,v)

defined by $u = \rho\cos\theta$, $v = \rho\sin\theta$ in the same nbd.

In the normal coordinates system,

(radial geodesic) $\theta_0 =$ constant $\Longleftrightarrow$ $(u, v) = t\,(u_0, v_0)$

In particular $\mathbf{X}(t, 0) = \alpha_{e_1}(t)$ & $t = \rho$

$$\Rightarrow \mathbf{X}_u(0,0) = \alpha'_{e_1}(0) = e_1$$

Similarly $\mathbf{X}(0, t) = \alpha_{e_2}(t)$ & $t = \rho$

$$\Rightarrow \mathbf{X}_v(0,0) = \alpha'_{e_2}(0) = e_2$$

$$\therefore \quad \left(g_{u^i u^j}(0,0) \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For other (u_0, v_0), we conclude that

$$u = t u_0 \ , \quad v = t v_0$$

satisfy the geodesic equations:

$(u^1, u^2) = (u, v)$

$$\frac{d^2u^k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{du^i}{dt}\frac{du^j}{dt} = 0$$

$$\Rightarrow \quad \boxed{\sum_{i,j} \Gamma^k_{ij}(p)\, u_0^i u_0^j = 0}$$

Since $(u_0^1, u_0^2) = (u_0, v_0)$ are arbitrary (as long as $u_0^2 + v_0^2 = 1$),

we have $\boxed{\Gamma^k_{ij}(p) = 0, \ \forall i, j, k.}$

This implies $\boxed{\left(\nabla_{X_{u^i}} X_{u^j}\right)(p) = 0, \quad \forall i, j.}$

$\left(\nabla_{X_{u^i}} X_{u^j} = \sum_k \Gamma^k_{ij} X_{u^k} ! \right)$

<u>Remark</u>: In modern Riemannian geometry, a coordinates

$X(u^1, u^2)$ around a point p satisfying

$$\begin{cases} g_{u^i u^j}(p) = \delta_{ij} \\ (\nabla_{X_{u^i}} X_{u^j})(p) = 0 \quad \left(\text{equivalently, } \Gamma^i_{jk}(p) = 0\right) \end{cases}$$

is called a <u>normal coordinates</u> at p.

Observe that

$$\frac{\partial}{\partial u^k} g_{u^i u^j} = \frac{\partial}{\partial u^k} \langle X_{u^i}, X_{u^j} \rangle$$

$$= \langle \nabla_{X_{u^k}} X_{u^i}, X_{u^j} \rangle + \langle X_{u^i}, \nabla_{X_{u^k}} X_{u^j} \rangle$$

$$\Rightarrow \boxed{\left.\frac{\partial}{\partial u^k} g_{u^i u^j}\right|_p = 0}$$

$$\therefore \boxed{g_{u^i u^j} = \delta_{ij} + 2^{\text{nd}} \text{ or higher order terms.}}$$

This shows the "curvature" is hidden in 2^{nd} order term of $g_{u^i u^j}$.

Now we can complete the prove of Gauss lemma.

By the change of coordinates formula

$$\sqrt{\det\begin{pmatrix} 1 & 0 \\ 0 & g_{22}(\rho,\theta)\end{pmatrix}} = \sqrt{\det(g_{u^i u^j})}\ \frac{\partial(u,v)}{\partial(\rho,\theta)} \qquad (\text{do Carmo})$$

where $\dfrac{\partial(u,v)}{\partial(\rho,\theta)} = \det\begin{pmatrix} \frac{\partial u}{\partial\rho} & \frac{\partial u}{\partial\theta} \\ \frac{\partial v}{\partial\rho} & \frac{\partial v}{\partial\theta}\end{pmatrix} = \det\begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta\end{pmatrix} = \rho$

$$\therefore \quad \sqrt{g_{22}(\rho,\theta)} = \rho\sqrt{\det(g_{u^i u^j})}$$

$$\Rightarrow \quad g_{22} \to 0 \qquad \left(\text{since } (g_{u^i u^j}) \to (\delta_{ij})\right)$$

& $\quad \left(\sqrt{g_{22}}\right)_\rho = \rho\left(\dfrac{\partial}{\partial\rho}\sqrt{\det(g_{u^i u^j})}\right) + \sqrt{\det(g_{u^i u^j})}$

$$\to 1 \qquad \left(\text{since } (g_{u^i u^j}) \text{ is smooth up to } \rho = 0\right)$$

$\#$

Gauss Curvature in Geodesic Polar Coordinates $\left(d\rho^2 + G\, d\theta^2 \right)$

Recall that, when $F = g_{12} = 0$, we have

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial\theta}\left(\frac{E_\theta}{\sqrt{EG}}\right) + \frac{\partial}{\partial\rho}\left(\frac{G_\rho}{\sqrt{EG}}\right)\right]$$

where $E = g_{11} = 1$, $G = g_{22}$.

$$\Rightarrow \quad K = -\frac{1}{2\sqrt{G}}\frac{\partial}{\partial\rho}\left(\frac{G_\rho}{\sqrt{G}}\right) = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial\rho}\left(\frac{G_\rho}{2\sqrt{G}}\right)$$

$$\therefore \quad \boxed{K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial\rho^2}}$$

i.e. $\boxed{\left(\sqrt{G}\right)_{\rho\rho} + K\sqrt{G} = 0}$. (Jacobi equation)

Applications of Geodesic Polar Coordinates:

(1) Surfaces of constant curvatures. Suppose M has constant curvature K. At any $p \in M$, we take a geodesic polar coordinates (ρ, θ) at p.

Then the metric has the form

$$d\rho^2 + G(\rho,\theta)\, d\theta^2 \quad \left(\text{i.e. } \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix} \right)$$

with
$$\begin{cases} \lim\limits_{\rho \to 0} \sqrt{G}(\rho,\theta) = 0 \\ \lim\limits_{\rho \to 0} (\sqrt{G})_\rho (\rho,\theta) = 1 . \end{cases}$$

By the formula for curvature, we have

$$(\sqrt{G})_{\rho\rho} + K\sqrt{G} = 0 .$$

Let $f = \sqrt{G}$, then

$$\begin{cases} f_{\rho\rho} + Kf = 0 . \quad \text{——} (1) \\ \lim\limits_{\rho \to 0} f(\rho,\theta) = 0 \\ \lim\limits_{\rho \to 0} f_\rho(\rho,\theta) = 1 \end{cases}$$

<u>Case 1</u> : $K = 0$.

(1) becomes $f_{\rho\rho} = 0$

$$\Rightarrow \quad f(\rho,\theta) = a(\theta)\rho + b(\theta) .$$

Moreover,

$$\begin{cases} 0 = \lim\limits_{\rho \to 0} f(\rho,\theta) = b(\theta) \\ 1 = \lim\limits_{\rho \to 0} f_\rho(\rho,\theta) = a(\theta) \end{cases}$$

$\therefore \quad f(\rho,\theta) = \rho$

That is $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$ $\quad \left(d\rho^2 + \rho^2 d\theta^2 \right)$

(The usual polar coordinates on $\mathbb{R}^2$.)

<u>Case 2</u> $\quad K > 0$.

$$f_{\rho\rho} + K f = 0$$

$$\Rightarrow \quad f(\rho,\theta) = A(\theta)\cos(\sqrt{K}\rho) + B(\theta)\sin(\sqrt{K}\rho)$$

$$\& \quad f_\rho(\rho,\theta) = -\sqrt{K}A(\theta)\sin(\sqrt{K}\rho) + \sqrt{K}B(\theta)\cos(\sqrt{K}\rho)$$

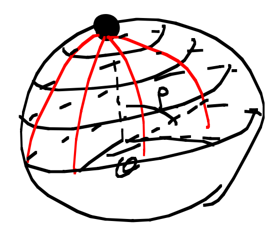
<u>Initial values</u> $\Rightarrow$

$$\begin{cases} 0 = A(\theta) \\ 1 = \sqrt{K}B(\theta) \end{cases} \Longrightarrow \quad B(\theta) = \frac{1}{\sqrt{K}}$$

Hence $f(\rho,\theta) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}\rho)$

$$\therefore \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{K}\sin^2(\sqrt{K}\rho) \end{pmatrix}$$

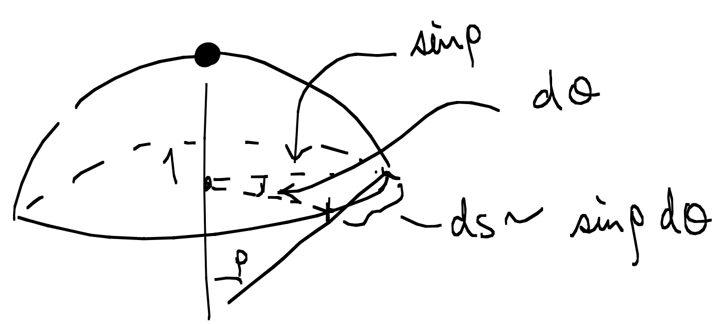
i.e. metric $= d\rho^2 + \frac{1}{K}\sin^2(\sqrt{K}\rho)\, d\theta^2$,



In particular for $K=1$, we have

$$d\rho^2 + \sin^2\rho\, d\theta^2,$$

the geodesic polar coordinates on S^2



<u>Case 3</u> $K < 0$.

$$f_{\rho\rho} + Kf = 0$$

$$\Rightarrow \quad f(\rho,\theta) = A(\theta)\cosh(\sqrt{-K}\,\rho) + B(\theta)\sinh(\sqrt{-K}\,\rho)$$

$$\& \quad f_\rho(\rho,\theta) = \sqrt{-K}A(\theta)\sinh(\sqrt{-K}\rho) + \sqrt{-K}B(\theta)\cosh(\sqrt{-K}\rho)$$

<u>Initial values</u> $\Rightarrow$

$$\begin{cases} 0 = A(\theta) \\ 1 = \sqrt{-K}\,B(\theta) \end{cases} \Rightarrow \quad B(\theta) = \frac{1}{\sqrt{-K}}.$$

$$\therefore \quad f(\rho,\theta) = \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}\,\rho)$$

i.e. $$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{(-K)}\sinh^2(\sqrt{-K}\rho) \end{pmatrix}$$

i.e. metric $= d\rho^2 + \frac{1}{(-K)} \sinh^2(\sqrt{-K}\rho)\, d\theta^2$

In particular, if $K=-1$, the metric

is $\quad d\rho^2 + \sinh^2\rho\, d\theta^2$

(the <u>hyperbolic</u> metric.)

<u>Conclusion</u>: Geodesic Polar Coordinates on a surface with constant curvature K are given by

$$d\rho^2 + (f_K(\rho))^2 d\theta^2$$

where

$$f_K(\rho) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}\rho), & K>0 \\ \rho, & K=0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}\rho), & K<0. \end{cases}$$

It is interesting to note that for $K > 0$,

$$f_K(\rho) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}\,\rho)$$

has zero at $\rho = \dfrac{\pi k}{\sqrt{K}}$, $k = 0, 1, 2, \cdots$.

Therefore, the metric $\quad d\rho^2 + f_K^2(\rho)\, d\theta^2$

is well-defined only up to $\rho < \dfrac{\pi}{\sqrt{K}}$. In this case, we called the point $\mathbf{x}\left(\frac{\pi}{\sqrt{K}}, \theta\right)$ is <u>conjugate</u> to p along the (radial) geodesic $\mathbf{x}(\rho, \theta)$. Note that in general, $G = G(\rho, \theta)$ and $\rho_0 > 0$ s.t. $G(\rho_0, \theta) = 0$ usually depends on θ, i.e. $\rho_0 = \rho_0(\theta)$. And we define

<u>Def</u>: • The "center point" $p = \mathbf{x}(0, \theta)$, $\forall \theta$, of a geodesic polar coordinate patch is call the <u>pole</u> of the patch.

- Let $X(\rho,\theta)$ = geodesic polar coordinate patch on M with pole p.

If $\tilde{p} = X(\rho_0,\theta)$ is point such that

$$\langle X_2, X_2\rangle = G(\rho_0,\theta) = 0,$$

then $\tilde{p}$ is called a <u>conjugate point</u> to p along the (radial) geodesic $X(\rho,\theta)$.

[<u>Remark</u>: Conjugate point along geodesic can be defined without refering to a geodesic polar coordinate. See do Carmo.]

Recall that in Ch5, we have shown that radial geodesic is length minimizing. In fact, we have the following:

<u>Thm</u> (Thm 6.8.10 of Oprea)

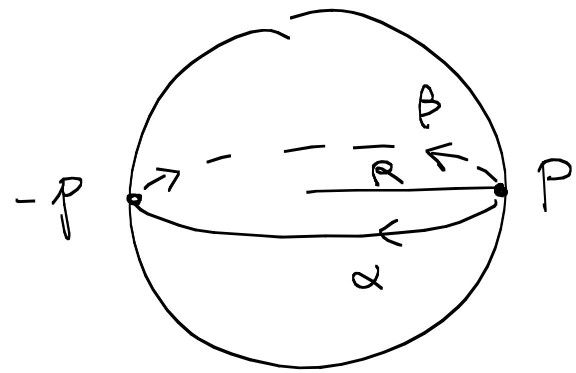
(1) If α is a geodesic joining $p \in M$ to $q \in M$ and there are no conjugate points to p along α between p and q, then α gives the shortest arclength of any curve which is close to α and which joins p & q.

(2) If $\hat{p} = \alpha(t_0)$ is conjugate to $p = \alpha(0)$ along the geodesic α, then α cannot give the shortest arclength among curves joining p to $q = \alpha(t_1)$ with $t_1 > t_0$.

(3) If M is geodesically complete, then any 2 points of M may be joined by a geodesic which has the shortest arclength among curves between the 2 points.

eg: Sphere S^2_R of radius.

Then $K = \frac{1}{R^2} \Rightarrow \sqrt{K} = \frac{1}{R}$.



Geodesic α from to p remains a shortest path before hitting $-p$.

ie. before length $\rho_0 = \pi R$,

which is exactly equal to $\frac{\pi}{\sqrt{K}}$.

Beyond $-p$ (ie $\rho > \frac{\pi}{\sqrt{K}}$) α is no longer minimizing since β from p opposite to α with have a shorter length.

Lemma: Suppose M is complete and $K \geq a^2 > 0$. Then any geodesic $\alpha : [0, \infty) \to M$ has a conjugate point somewhere in the interval $(0, \pi/a]$.

[Note: M complete $\Rightarrow$ $\forall$ geodesic defines on $(-\infty, \infty)$
In particular, geodesic polar coordinates $\mathbf{X}(\rho, \theta)$
defines on $0 < \rho < +\infty$, $0 < \theta < 2\pi$.]

Pf: Let $\mathbf{X}(\rho, \theta)$ = geodesic polar coordinate with pole $\alpha(0)$,

and $\begin{pmatrix} 1 & 0 \\ 0 & f^2 \end{pmatrix}$ be the metric coefficients of $\mathbf{X}$,
$(f \geq 0)$

Then
$$\begin{cases} f_{\rho\rho} + Kf = 0. & \text{——(1)} \\ \lim_{\rho \to 0} f(\rho, \theta) = 0 \\ \lim_{\rho \to 0} f_\rho(\rho, \theta) = 1 \end{cases}$$

Since $f \geq 0$, we have

$$f_{\rho\rho} + a^2 f \leq 0$$

Let $\quad h = \sin(a\rho) f_\rho - af\cos(a\rho).$

Then
$$h_\rho = \sin(a\rho) f_{\rho\rho} + a\cos(a\rho) f_\rho - a f_\rho \cos(a\rho) + a^2 f \sin(a\rho)$$

$$= \sin(a\rho)(f_{\rho\rho} + a^2 f)$$
$$\leq 0 \qquad \text{for } 0 \leq \rho \leq \frac{\pi}{a}.$$

$\Rightarrow$
$$\sin(a\rho) f_\rho - af\cos(a\rho) \leq \lim_{\rho\to 0} \sin(a\rho) f_\rho - af\cos(a\rho)$$
$$= 0 \qquad \text{for } 0 \leq \rho \leq \frac{\pi}{a}$$

$\Rightarrow$
$$\left(\frac{f}{\sin(a\rho)}\right)_\rho = \frac{\sin(a\rho) f_\rho - f a\cos(a\rho)}{\sin^2(a\rho)} \leq 0 \qquad \text{for } 0 < \rho < \frac{\pi}{a}$$

$$\Rightarrow \qquad \frac{f}{\sin(a\rho)} \le \lim_{\rho \to 0} \frac{f}{\sin(a\rho)}$$

Note that $\displaystyle \lim_{\rho\to 0} \frac{f}{\sin(a\rho)} = \lim_{\rho \to 0} \frac{f_\rho}{a\cos a\rho} = \frac{1}{a}$

$$\therefore \qquad \frac{f}{\sin(a\rho)} \le \frac{1}{a} \qquad \forall \quad 0 < \rho < \frac{\pi}{a}$$

$$\Rightarrow \qquad f \le \frac{1}{a}\sin(a\rho) \qquad \forall\ 0 \le \rho < \frac{\pi}{a}$$

Hence $\quad f(\frac{\pi}{a}, \theta) \le 0.$

$\Rightarrow \quad f(\rho, \theta) = 0$ for some $\rho \le \frac{\pi}{a}$. $\#$

This lemma $\Rightarrow$ any shortest geodesic α on complete M with $K \ge a^2 > 0$ has length $\le \frac{\pi}{a}$.

Therefore, the diameter $\mathrm{Diam}(M)$ of M defined by

$$\mathrm{Diam}(M) = \sup\{ d(p,q) : p \,\&\, q \in M \}$$

must satisfies $\mathrm{Diam}(M) \le \frac{\pi}{a}$.

(by Thm 6.8.10 of the text book.) Hence we have

Thm (Bonnet, Thm 6.8.16 of the text book)

Suppose M is complete and $K \ge a^2 > 0$. Then $\mathrm{Diam}(M) \le \pi/a$ and consequently, M is compact.